A GENERALIZED KOSZUL COMPLEX

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1. This note is concerned with the definition of a free complex which generalizes the classical Koszul complex [2], and its application to the notions of depth and multiplicity.

Let $f\colon R^m \to R^n$ be a map, where R is any commutative ring. Then for each p, $1 \leqq p \leqq n$, we define a complex K as follows:

$$K_{q+2} = \sum \bigwedge^{s_0+\nu} R^{n*} \otimes \bigwedge^{s_1} R^{n*} \otimes \cdots \otimes \bigwedge^{s_q} R^{n*} \otimes \bigwedge^{n+1+\Sigma s_i} R^m \quad \text{for } q \geqq 0;$$

$$K_1 = \bigwedge^p R^m, \qquad K_0 = \bigwedge^p R^n;$$

ν is the fixed integer $n+1-p$ and the summation is taken over all $s_0 \geqq 0$, $s_i \geqq 1$ for $i \geqq 1$. (We are using the notation $M^* = \operatorname{Hom}(M, R)$ for any R-module M.) The boundary map in this complex is defined as

$$\begin{aligned} d_{q+2}&(b_0 \otimes b_1 \otimes \cdots \otimes b_q \otimes a) \\ &= \sum_{i=0}^{q-1} (-1)^i b_0 \otimes \cdots \otimes b_i \wedge b_{i+1} \otimes \cdots \otimes b_q \otimes a \\ &\quad + (-1)^q b_0 \otimes \cdots \otimes b_{q-1} \otimes \omega_{b_q}(a) \qquad \text{for } q \geqq 0, \end{aligned}$$

and $d_1\colon \bigwedge^p R^m \to \bigwedge^p R^n$ is simply $\bigwedge^p f$. The symbol $\omega_{b_q}(a)$ is defined as follows: if β is any element of R^{n*}, then βf is in R^{m*} and thus induces a derivation of degree -1 on the exterior algebra of R^{m*}, denoted by ω_β. If $b = \beta_1 \wedge \cdots \wedge \beta_s \in \bigwedge^s R^{n*}$ and $a \in \bigwedge^t R^m$, then $\omega_b(a)$ is defined to be $\omega_{\beta_1} \cdots \omega_{\beta_s}(a)$. Since $\omega_{\beta_1}\omega_{\beta_2} + \omega_{\beta_2}\omega_{\beta_1} = 0$, this operation is well defined and thus gives a pairing $\bigwedge^s R^{n*} \otimes \bigwedge^t R^m \to \bigwedge^{t-s} R^m$.

The fact that this gives a chain complex is rather easy to verify and the length of the complex is seen to be $m-n+1$.

2. If R is a commutative, noetherian ring, and E is an R-module, a sequence $a_1, \cdots, a_d$ of elements in R is called a proper E-sequence if for all i, $1 \leqq i \leqq d$, $E/(a_1, \cdots, a_i)E \neq 0$ and a_i is not a zero-divisor for $E/(a_1, \cdots, a_{i-1})E$ [1]. If I is an ideal of R, the I-depth of E is defined to be the length of a maximal proper E-sequence of elements contained in I. It is known that this number is always finite for a

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noetherian ring R, and that any two such maximal proper E-sequences have the same length [1].

If $f: R^m \rightarrow R^n$, the ideal $I(f)$ is defined to be the annihilator of Coker $\wedge^n f$. If E is an R-module, we denote the homology of $K \otimes E$ by $H_*(\wedge^p f, E)$, and that of $\operatorname{Hom}(K, E)$ by $H^*(\wedge^p f, E)$.

THEOREM 1. *Given a map $f: R^m \rightarrow R^n$ ($m \geqq n$, and R a noetherian ring) and an R-module E such that $E/I(f)E \neq 0$, we have, for each p ($1 \leqq p \leqq n$) the following statements*:

(1) *$I(f)$-depth E is the smallest integer q for which $H^q(\wedge^p f, E) \neq 0$, and $H^d(\wedge^p f, E) = \operatorname{Ext}^d(\text{Coker } \wedge^p f, E)$, where $d = I(f)$-depth E.*

(2) *$m-n+1-(I(f)$-depth $E) =$ the largest integer q for which $H_q(\wedge^p f, E) \neq 0$. Furthermore, $H_{m-n+1-d}(\wedge^p f, E)$ may also be interpreted as $\operatorname{Ext}^d(M, E)$ where M is a specific module depending upon f and p.*

The proof of this theorem follows from a fairly general argument about exact connected sequences of functors $\{T^i\}$ satisfying several conditions, the main one being that for every R-module E, Supp $T^i(E) \subset R/I$ for some fixed ideal I. It can be shown that $H^*(\wedge^p f, E)$ and $H_*(\wedge^p f, E)$ (the latter with suitable shift of index) are both exact connected sequences of functors of the appropriate type, and hence our result.

As a corollary, we obtain the fact that if E and f are as above, then $I(f)$-depth $E \leqq m-n+1$. We also obtain the generalized Cohen-Macaulay unmixedness theorem, due to Eagon [3] which is:

THEOREM 2. *Let $f: R^m \rightarrow R^n$ ($m \geqq n$) be a map such that $I(f)$-depth $R = m-n+1$. Then if R is a Cohen-Macaulay ring,* Coker *$\wedge^p f$ is unmixed for all p, $i \leqq p \leqq n$.*

The proof proceeds by using the fact that if $I(f)$-depth $R = m-n+1$, then $K_{\wedge^p f}$ is an acyclic resolution of Coker $\wedge^p f$ and $K^*_{\wedge^p f} = \operatorname{Hom}(K_{\wedge^p f}, R)$ is an acyclic resolution of $H^{m-n+1}(\wedge^p f, R)$. Thus $\operatorname{Ext}^i(H^{m-n+1}(\wedge^p f, R), R)$ is 0 if $i < m-n+1$, and is Coker $\wedge^p f$ if $i = m-n+1$. Thus one may express Coker $\wedge^p f$ as $\operatorname{Ext}^{m-n+1}(H^{m-n+1}(\wedge^p f, R), R)$ which is equidimensional. Since R is Cohen-Macaulay, this implies that Coker $\wedge^p f$ is unmixed.

3. Let us assume throughout that R is a local ring,[2] unless otherwise specified. Although this is not an essential assumption, it simplifies some of our statements.

If M is an R-module, we denote by $S(M) = \sum S_\nu(M)$ the symmetric

[2] Here, by a local ring, we mean a commutative, noetherian ring with identity element, having a unique maximal ideal.

algebra generated by M over R. If $f\colon R^m \to R^n$ is a map, then $S_\nu(f)\colon S_\nu(R^m) \to S_\nu(R^n)$ is the induced map on symmetric products. If, moreover, E is an R-module such that $\operatorname{Coker} f \otimes E$ has finite length, then the same is true for $\operatorname{Coker} S_\nu(f) \otimes E$ for every $\nu \geqq 1$, i.e., $l(\operatorname{Coker} S_\nu(f) \otimes E) < \infty$. We define a numerical function $P_f(\nu, E) = l(\operatorname{Coker} S_\nu(f) \otimes E)$.

For any polynomial function ϕ, we define $\mu(\phi) = (\deg \phi)!$ (leading coefficient of ϕ).

THEOREM 3. *Let R be a local ring, $f\colon R^m \to R^n$ be a map, and E be an R-module such that $l(\operatorname{Coker} f \otimes E) < \infty$. Then*

(1) *$P_f(\nu, E)$ is a polynomial function for all sufficiently large ν;*

(2) *$\mu(P_f(\nu, E))$ depends only on* Coker *f and E;*

(3) $\deg P_f(\nu, E) = n - 1 + \dim E$.

The proof, although not trivial, is computational. Let $\tau^m\colon S(R^m)^m \to S(R^m)$ be the canonical map. $S(R^n)$, and hence also $S(R^n) \otimes E$, becomes a graded $S(R^m)$-module through the map $S(f)\colon S(R^m) \to S(R^n)$. Hence we may consider the Koszul complex K_{τ^m} associated with the map τ^m, and the (graded) homology groups $H_*(\tau^m, S(R^n) \otimes E)$ whose λth homogeneous part is denoted by $H_*(\tau^m, S(R^n) \otimes E)_\lambda$. We then observe that $\Delta^m P_f(\nu, E) = \sum (-1)^{m-q} l(H_{m-q}(\tau^m, S(R^n) \otimes E)_{\nu+q})$ for all sufficiently large ν, where $\Delta^m P_f$ denotes the mth difference function of P_f, and establishes, through the use of a certain double complex, that $\Delta^{m+n} P_f(\nu, E) = 0$ for all sufficiently large ν. This establishes (1).

The proof of (2) is a purely formal computation, based on an application of Schanuel's lemma, which also shows that $\deg P_f(\nu, E) - n + 1$ depends only on $\operatorname{Coker} f$ and E.

Having proved these facts, the proof of (3) proceeds by restricting our attention to maps $f\colon R^m \to R^n$ such that $f(R^m) \subset \mathfrak{m} R^n$, where $\mathfrak{m}$ is the maximal ideal of R. Since, then, $I(f) \subset \mathfrak{m}$, $\deg P_f - n + 1$ gets squeezed between the degree of the characteristic function for $\mathfrak{m}$ and that for $I(f)$ with respect to E. Hence the result.

As a consequence of this theorem, we obtain a generalization of Krull's principal ideal theorem.

THEOREM 4. *Let R be any noetherian ring, and $f\colon R^m \to R^n$, with $m \geqq n$. Then* $\dim R_{\mathfrak{p}} \leqq m - n + 1$ *for all minimal primes $\mathfrak{p}$ in* Supp Coker *f.*

The proof proceeds by first reducing to the case when R is a local ring and $\operatorname{Coker} f$ has finite length. One then may assume that R is an integral domain and thus contains a nonzero divisor α in its maximal ideal. From the exact sequence $0 \to S(R^n) \to^{\alpha} S(R^n) \to S(R^n) \otimes R/\alpha \to 0$ (the maps being of homogeneous degree 0), and using the fact that $\Delta^m P_f$ may be computed as an Euler-Poincaré characteristic, together with Theorem 3, we obtain our result.

We also obtain as a corollary (again R is a local ring) that if $f: R^m \rightarrow R^n$ and E is such that $l(\text{Coker } f \otimes E) < \infty$, then $m-n+1 \geqq \dim E$ and hence $m \geqq \deg P_f(\nu, E)$.

These theorems naturally lead to the following definition. Let M be a module of finite length over a local ring R, let $f: R^m \rightarrow R^n$ be a map whose cokernel is M, and let E be an R-module. Then $(\dim R+n-1)!$ (the coefficient of the term of degree $n-1+\dim R$ in the polynomial $P_f(\nu, E)$) is a non-negative integer which depends only on M and E. We call it the multiplicity of M with respect to E, and denote it by $e_E(M)$.

THEOREM 5. *If R is a local ring, given a map $f: R^m \rightarrow R^n$ and an R-module E such that $l(\text{Coker } f \otimes E) < \infty$, we have*

$$\binom{n-1}{n-p} \Delta^m P_f(\nu, E) = \chi H_*\left(\bigwedge^p f, E\right)$$

where $\chi H_*(\bigwedge^p f, E) = \sum_q (-1)^q l(H_q(\bigwedge^p f, E))$.

The proof of this theorem falls out of a general construction of a double complex associated with maps $R^m \rightarrow^f R^n \rightarrow^g R^r$ which relates the complexes K defined by f, g, and gf.

We say the map $f: R^m \rightarrow R^n$ is a parameter matrix for R if $M = \text{Coker} f$ has finite length, and $m-n+1 = \dim R$. We obtain as a corollary of the above theorem that if $f: R^m \rightarrow R^n$ is a parameter matrix for R with cokernel M, and if E is an R-module, then

$$\chi H_*\left(\bigwedge^p f, E\right) = \binom{n-1}{n-p} e_E(M).$$

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