ON SOME SEMI-GROUPS

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THEOREM 1. *Let Φ be an infinite field, $f(x, y)$ a polynomial in x, y over Φ. Let α, β be any elements in Φ. In order that Φ may be a semi-group under the operation $\alpha \circ \beta = f(\alpha, \beta)$, the possible types of $f(x, y)$ are the following:*

$$axy + b(x + y) + \frac{b(b - 1)}{a}, \qquad x + y + d, \qquad d, \qquad x, \qquad y,$$

where a, b, d denote any constants in Φ, and $a \neq 0$.

Moreover under these operations, Φ is respectively a commutative group with zero, a commutative group, a trivial semi-group (all composites are zero), an "anti-semi-groupe à droite" and an "anti-semi-groupe à gauche."

In the proof of this theorem, after we arrange lexicographically the terms of $f(x, y)$, and compare the leading terms of $(x \circ y) \circ z$ and $x \circ (y \circ z)$, we find that $f(x, y)$ does not include terms of degree $\geqq 3$. Next, from the relations between the coefficients which express that $(x \circ y) \circ z = x \circ (y \circ z)$, we derive five types as a conclusion. Further we obtain Theorem 1 by investigating their structures. Moreover using the result of Theorem 1, we obtain

THEOREM 2. *Let Φ be an infinite field, $f(x, y)$, $g(x, y)$ polynomials over Φ. Then, in order that Φ be an associative semi-ring under the operations $x \oplus y = f(x, y)$, $x \otimes y = g(x, y)$, the possible types are limited to six. Of them, Φ is a field under*

$$x \oplus y = x + y + \frac{b'}{a'}, \quad x \otimes y = a'xy + b'(x + y) + \frac{b'(b' - 1)}{a'} \qquad (a' \neq 0);$$

and Φ is a trivial ring under

$$x \oplus y = x + y + d, \qquad x \otimes y = -d.$$

Further, as an application of this theorem, we can see that there exists no lattice with the defining polynomial operations:

$$x \cup y = f(x, y), \qquad x \cap y = g(x, y).$$

On the other hand, we lay down the following

DEFINITION. Let $\overline{S}=\{s_1, s_2, \cdots\}$ be a semi-group, and consider the set $S_i=\{s_i, s_i', s_i'', \cdots, s_i^{(m)}, \cdots\}$, obtained by joining a finite or an infinite number of elements

$$s_i', s_i'', \cdots, s_i^{(m)}, \cdots$$

to every element s_i. If we define products in

$$S = \{s_1, s_1', \cdots, s_2, s_2', \cdots, \cdots\}$$

as follows:

$$s_i^{(m)} \circ s_j^{(n)} = s_i \circ s_j,$$

S becomes a semi-group, which we shall call the "prolongation" of $\overline{S}$. Then we have

THEOREM 3. *Let Φ be an infinite field and let $a, b, c, d, k; a', b', c', d', k'$ be arbitrary constants in Φ. When we define a new operation by*

$$(x, y) \circ (z, u) = (ax + by + cz + du + k,\ a'x + b'y + c'z + d'u + k'),$$

there exist only 22 *types such that $\Phi\times\Phi$ forms a semi-group. They are:*

(1°) *an anti-semi-groupe à droite*: $(x, y) \circ (z, u) = (x, y)$.

(2°) *an anti-semi-groupe à gauche*: $(x, y) \circ (z, u) = (z, u)$.

(3°) *a trivial semi-group*: $(x, y) \circ (z, u) = (k, k')$.

(4°) *a commutative group*: $(x, y) \circ (z, u) = (x+z+k,\ y+u+k')$.

(5°) *prolongations of the commutative groups*:

$$(x, y) \circ (z, u) = (x + z + k,\quad a'(x + z) + k'),$$

$$(x, y) \circ (z, u) = (k,\quad a'(x + z) + (y + u) + k'),$$

$$(x, y) \circ (z, u) = \left(a(x + z) + b(y + u) + k,\ \left(\frac{1 - a}{b}\right)[a(x + z) + b(y + u)] + k'\right).$$

(6°) *idempotent semi-groups*:

$$(x, y) \circ (z, u) = \left(a(x - z) + b(y - u) + z,\ \left[\frac{a(1 - a)}{b}\right](x - z) - a(y - u) + y\right),$$

$$(x, y) \circ (z, u) = (x,\quad a'(x - z) + u),$$

$$(x, y) \circ (z, u) = (z,\quad a'(x - z) + y).$$

(7°) *prolongations of the anti-semi-groupes à gauche*:

$$(x, y) \circ (z, u) = (z, \quad c'z + k'),$$
$$(x, y) \circ (z, u) = (k, \quad c'(z - k) + u),$$
$$(x, y) \circ (z, u) = \left(cz + du + k, \quad \left(\frac{1 - c}{d}\right)(cz + du) - \frac{ck}{d}\right).$$

(8°) *prolongations of the anti-semi-groupes à droite*:

$$(x, y) \circ (z, u) = (x, \quad a'x + k'),$$
$$(x, y) \circ (z, u) = (k, \quad a'(x - k) + y),$$
$$(x, y) \circ (z, u) = \left(ax + by + k, \quad \left(\frac{1 - a}{b}\right)(ax + by) - \frac{ak}{b}\right).$$

(9°) *unions of infinite left ideals*:

$$(x, y) \circ (z, u) = ((x + k) + z, \quad a'(x + k) + u),$$
$$(x, y) \circ (z, u) = (z, \quad a'x + (y + u) + k'),$$
$$(x, y) \circ (z, u) = \Big((ax + by + k) + z, \quad \left(\frac{1 - a}{b}\right)(ax + by + k) + u\Big).$$

(10°) *unions of infinite right ideals*:

$$(x, y) \circ (z, u) = ((z + k) + x, \quad c'(z + k) + y),$$
$$(x, y) \circ (z, u) = (x, \quad c'z + (y + u) + k'),$$
$$(x, y) \circ (z, u) = \Big((cz + du + k) + x, \quad \left(\frac{1 - c}{d}\right)(cz + du + k) + y\Big).$$

b, d denote nonzero constants.

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