A BORDISM THEORY FOR ACTIONS OF AN ABELIAN GROUP

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1. Introduction. This note is a preliminary sketch of a general bordism theory for the differentiable actions of a finite abelian group on closed manifolds. The present note is based upon the techniques outlined in [1] for the study of differentiable periodic maps. We fix a finite abelian group A and in A we choose a family K of subgroups. We assume that any subgroup of an element in K is also an element in K. We wish to consider all differentiable actions (A, B^n) on compact manifolds (possibly with boundary) which have the property that each isotropy group A_x is an element of K. Two such actions are strictly equivalent if and only if they are connected by an equivariant diffeomorphism.

We now describe the equivariant bordism theory. An action (A, M^n) on a closed manifold, all of whose isotropy groups lie in K, is said to equivariantly bord if and only if there is an (A, B^{n+1}), all of whose isotropy groups also belong to K, for which the induced action on the boundary $(A, \partial B^{n+1})$ is equivariantly diffeomorphic to (A, M^n). From two actions (A, M_1^n) and (A, M_2^n) a disjoint union action may be formed $(A, M_1^n \cup M_2^n)$ with $M_1^n \cap M_2^n = \emptyset$, and with A restricted to M_i^n equal to (A, M_i^n) for $i=1, 2$. We shall say that (A, M_1^n) is equivariantly bordant to (A, M_2^n) if and only if their disjoint union equivariantly bords. Again we recall that every isotropy group is to be a member of the family K. We have defined an equivalence by introducing the equivariant bordism relation. The proof of transitivity is based on an equivariant collaring theorem which asserts that for any differentiable (A, B^{n+1}) there is an open invariant $U \supset \partial B^{n+1}$ and an equivariant diffeomorphism $m: (A, U) \to (A, \partial B^{n+1} \times [0, 1))$ for which $m(x) = (x, 0)$, $x \in \partial B^{n+1}$, and where $(A, \partial B^{n+1} \times [0, 1))$ is given by $\alpha(x, t) = (\alpha(x), t)$. We denote the unoriented bordism class of (A, M^n) by $[A, M^n]_2$ and the collection of all such equivalence classes by $I_n(A; K)$. An abelian group structure, in which every element has order 2, can be imposed on $I_n(A; K)$. We shall exhibit the basic fact that this is a finite group. On the weak direct sum $I_*(A; K) = \sum_0^\infty I_n(A; K)$ we can impose a graded right module structure over the unoriented Thom bordism ring $\mathfrak{N}$. For

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each (A, M^n) and each closed V^m we define $(A, M^n \times V^m)$ by $\alpha(x, y) = (\alpha(x), y))$. We define the $\mathfrak{N}$-module structure by $[A, M^n]_2[V^m]_2 = [A, M^n \times V^m]_2$.

2. **Groups of bundle maps.** We consider a fibre bundle $[B, X, Y, G; \pi]$ with structure group G, a compact Lie group, which acts effectively from the left on the fibre Y. We wish to consider an action of A on $[B, X, Y, G; \pi]$ as a group of bundle maps. This means there are actions (A, B) and (A, X) for which $\pi: B \to X$ is equivariant. In addition each $\alpha \in A$ is a bundle map of $[B, X, Y, G; \pi]$ in the sense of [**2**, p. 9]. We have studied the situation presented here in the detailed exposition of the results announced in [**1**], however we use here a basic approach suggested by Samelson.

We can interpret the action of A on $[B, X, Y, G; \pi]$ as follows. Let $W \to X$ be the principal bundle, then G acts freely on the right of W as the group of right bundle translations to give the right principal G-space (W, G). The action of A as a group of bundle maps is then translated into a left action of A on W as a group of G-equivariant maps. We denote the resulting object by $(A, (W, G))$. Let $H \subset A$ be the subgroup of elements which map every orbit of (W, G) into itself. At each $x \in W$ we define a homomorphism $r_x: H \to G$ as follows. For each $h \in H$ there is a unique $g_h \in G$ with $h(x) = x \cdot g_h$. We set $r_x(h) = g_h$. Using G-equivariance, $h_1(x \cdot g_{h_2}) = h_1(x) \cdot g_{h_2} = x \cdot g_{h_1} g_{h_2}$ so that r_x is a homomorphism. We note that for $g \in G$, $r_{xg}(h) = g^{-1} r_x(h) g$. We shall assume that in fact for any pair of points x, y in W that r_x is conjugate in G to r_y. If $W/G = X$ is connected, then this condition is automatically satisfied.

For each $x \in W$ we set $S_x = \{y / r_y = r_x\}$. This is a closed subset of W which meets each orbit of (W, G). In addition $S_x \cap S_x g \neq \emptyset$ if and only if $g \in C(H_x)$, the centralizer of the image of r_x in G. We obtain thus a right principal space $(S_x, C(H_x))$ with $S_x / C(H_x) = X$. We may use S_x to define a cross-section of the associated $G/C(H_x)$-bundle, $(W \times G/C(H_x))/G \to X$ and to thereby obtain a reduction of the structure group to $C(H_x)$.

We fix A, a subgroup H, and a homomorphism $r: H \to G$. We consider all objects $(A, (W, G))$ where

(i) for any $x \in W$ the homomorphism r_x is conjugate in G to r,

(ii) if $\alpha \in A$ maps one orbit of (W, G) into itself then α carries every orbit into itself and $\alpha \in H$.

We note that $\pi: W \to W/G = X$ naturally induces an action (A, X) in which H is the subgroup leaving every point of X fixed. The condition (ii) is equivalent to requiring A/H act freely on X. Two such objects $(A, (W, G))$ and $(A, (W_1, G))$ are equivalent if and only if

W and W_1 are connected by a homeomorphism which is both A and G-equivariant.

Let $S(r)=\{y/y\in W,\ r_y=r\}$ and let $C(r)\subset G$ be the centralizer of the image of r. The product $A\times C(r)$ acts on $S(r)$ by $y\cdot(\alpha, g)=\alpha^{-1}(y)\cdot g$. The subgroup $\Delta(r)=\{h, r(h)\}$, $h\in H$, acts trivially on $S(r)$, thus an action of $L(r)=A\times C(r)/\Delta(r)$ is induced. This action is free, for if $y\cdot(\alpha, g)=\alpha^{-1}(y)\cdot g=y$, then $\alpha\in H$ and $r(\alpha)=g$. The original object $(A, (W, G))$ can be recovered completely from the right principal space $(S(r), L(r))$.

(2.1) *The equivalence classes of those objects* $(A, (W, G))$ *which contain a point* x *at which* $r_x=r$ *is in natural* 1-1 *correspondence with the equivalence classes of right principal* $L(r)$*-spaces.*

We again emphasize the assumption that A/H acts freely on X. The quotient space of $(S(r), L(r))$ is X/A. We are now in a position to define a bordism theory for differentiable objects $(A, (W, G))$ with $H\subset A$ fixed and $r: H\to G$ fixed. Here W is a compact differentiable manifold on which A and G act differentiably. The reader may define the appropriate bordism relation. We set $\dim\,[A, (W, G)]_2=\dim W/G$, and denote the resulting bordism group by $A_n(r: H\to G)$. Let $B(L(r))$ be the classifying space of $L(r)$. In view of (2.1) we have

(2.2) *The bordism module* $A_*(r: H\to G)$ *is naturally isomorphic to* $\mathfrak{N}_*(B(L(r)))$.

The bordism module of the space $B(L(r))$ was defined in [1] where it was noted that $\mathfrak{N}_*(B(L(r))\simeq H_*(B(L(r)); Z_2))\otimes\mathfrak{N}$. We are especially concerned with $G=O(k)$, the orthogonal group, and in *admissible* representations $r: H\to O(k)$. A representation is admissible if and only if the induced action of H on R^k has the O-vector as its only stationary point. We let $A_n(H\to O(k))=\sum_j A_n(r_j: H\to O(k))$ where the sum is taken over conjugacy classes of admissible representations of H in $O(k)$.

3. **The exact sequence.** We return to $I_n(A; K)$. The family K is partially ordered by inclusion. We let $H\in K$ be a maximal element of K and we let D be the family of subgroups of A obtained by deleting H from K. Note that we do not delete the proper subgroups of H. A subgroup of an element in D is also in D since H was maximal. Obviously there is a natural homomorphism $i_*: I_n(A; D)\to I_n(A; K)$. We next define a homomorphism $j_*: I_n(A; K)\to\sum_0^n A_{n-k}(H\to O(k))$. We consider a differentiable action (A, M^n) on a closed manifold with all isotropy groups in K. We let $F\subset M^n$ be the set of stationary points of H. This F is the finite disjoint union of closed connected regular submanifolds of M^n. At each $x\in F$ the isotropy subgroup A_x is H, thus A/H acts freely on F. Let F^{n-k} be the union of the $(n-k)$-

dimensional components of F. We may regard A as a group of Riemannian isometries so A acts as a group of bundle maps on the normal bundle to F^{n-k}. The subgroup H sends each normal fibre into itself with only the O-vector as stationary point. We may thus group the components of F^{n-k} according to the conjugacy class of the admissible representation of H on the normal fibres. To each such grouping we assign the appropriate element of $A_{n-k}(r_j\colon H\to O(k))$. In this way j_* is defined. We shall agree that $A_n(H\to O(0))=\mathfrak{N}_n(B(A/H))$, where $B(A/H)$ is the classifying space of A/H. We assign to F^n the bordism class of the principal A/H-bundle $F^n\to F^n/A$.

Next we define $\partial_*\colon \sum_0^n A_{n-k}(H\to O(k))\to I_{n-1}(A;\, D)$. We consider an object $(A,\,(W,\,O(k)))$ with $W/O(k)=V^{n-k}$. We form a $(k-1)$-sphere bundle by letting $O(k)$ act on $W\times S^{k-1}$ via $g(x,\,y)=(xg^{-1},\,gy)$ and the passing to $(W\times S^{k-1})/O(k)\to V^{n-k}$. The group A acts on $(W\times S^{k-1})/O(k)$ by $\alpha(x,\,y)=(\alpha(x),\,y)$. Since we are concerned only with admissible representations of H it follows that each isotropy group in $(A,\,(W\times S^{k-1})/O(k))$ is a proper subgroup of H. Thus we consider $[A,\,(W\times S^{k-1})/O(k)]_2\in I_{n-1}(A;\,D)$. This defines the boundary $\partial_*\colon\sum A_{n-k}(H\to O(k))\to I_{n-1}(A;D)$. We agree that $\partial_*(A_n(H\to O(0)))=0$.

(3.1) *The sequence*

$$\cdots\to I_n(A;D)\xrightarrow{i_*}I_n(A;K)\xrightarrow{j_*}\sum A_{n-k}(H\to O(k))\xrightarrow{\partial_*}I_{n-1}(A;D)\to\cdots$$

is exact.

The proof is entirely geometric. As a corollary we obtain

(3.2) *For every $n\geqq 0$ and every family K the group $I_n(A;\,K)$ is finite.*

We observe that $\sum_0^n A_{n-k}(H\to O(k))$ is finite. If $K=\{0\}$, then $I_n(A;K)=\mathfrak{N}_n(B(A))\simeq\sum H_{n-j}(B(A);\,Z_2)\otimes\mathfrak{N}_j$. We can now use (3.1) to prove (3.2) by induction on the number of elements in K.

This completes our outline. Later we shall take up special applications as well as the corresponding oriented groups.

References

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