IDEMPOTENTS IN GROUP ALGEBRAS

BY WALTER RUDIN[1]

Communicated November 26, 1962

I. **Introduction.** If G is a group, its group algebra $L^1(G)$ consists of all complex functions f on G for which the norm

$$\|f\| = \sum_{x \in G} |f(x)| \tag{1}$$

is finite; addition is pointwise, and multiplication is defined by convolution:

$$(f * g)(x) = \sum_{y \in G} f(y) g(y^{-1}x). \tag{2}$$

Any $f \in L^1(G)$ for which

$$f * f = \tag{3}$$

will be called an *idempotent on G*.

The *support* of a complex function f on G is the set of all $x \in G$ at which $f(x) \neq 0$. The *support group* of f is the smallest subgroup of G which contains the support of f.

By methods involving Fourier transforms and the Pontryagin duality theory, the idempotents on abelian groups are completely known [**2**, p. 199]. (For nondiscrete locally compact abelian groups, the classification of the idempotent measures was completed by P. J. Cohen [**1**].) Let us draw attention to the following facts, of which (A) and (D) are probably the most striking:

(A) If f is an idempotent on an abelian group G, *then the support group of f is finite.*

(B) Idempotents on abelian groups are *self-adjoint* (i.e., $f(x^{-1})$ is the complex conjugate of $f(x)$).

(C) On a finite abelian group there are only *finitely many* idempotents (namely 2^n if the group has n elements). On a countable abelian group there are *at most countably many* idempotents.

(D) If f is an idempotent on an abelian group and if $\|f\| > 1$, then $\|f\| \geqq \frac{1}{2}\sqrt{5}$ [**3**, p. 72]. (Note that there are no idempotents f with $\|f\| < 1$, except $f = 0$.)

It is the purpose of the present note to show that each of the above statements becomes false if the word "abelian" is omitted.

[1] Sponsored by the Mathematics Research Center, United States Army, Madison, Wisconsin under Contract No. DA-11-022-ORD-2059, and by NSF grant GP-249.

II. Consider a set E which contains the integers and the three symbols α, β, γ, let

$$(4)\qquad \begin{aligned} a &= (\alpha\beta\gamma), \\ b &= (\beta\gamma)(\cdots -2\ -1\ 0\ 1\ 2 \cdots) \end{aligned}$$

be permutations of E, and let G be the group generated by a and b. The relations

$$(5)\qquad a^3 = 1, \qquad b^{2k-1}a = a^2b^{2k-1}$$

hold for all integers k, and G consists of the distinct elements

$$(6)\qquad a^nb^k \qquad (n = 0, 1, 2;\ k = 0, \pm 1, \pm 2, \cdots).$$

Setting $\omega = \exp\{2\pi i/3\}$, define

$$(7)\qquad f_0(a^nb^k) = \begin{cases} \frac{1}{3}\omega^n & \text{if } k = 0, \\ 0 & \text{if } k \neq 0, \end{cases}$$

and

$$(8)\qquad f_j(x) = f_0(xb^{-j}) \qquad (x \in G;\ j = 0, \pm 1, \pm 2, \cdots).$$

I claim that

$$(9)\qquad f_0 * f_j = f_j \quad \text{and} \quad f_{2m-1} * f_j = 0$$

for all integers j and m. Indeed,

$$\begin{aligned} (f_0 * f_j)(a^nb^j) &= \sum_{r=0}^{2} f_0(a^{n-r})f_j(a^rb^j) \\ &= \frac{1}{9}\sum_{r=0}^{2} \omega^{n-r}\cdot\omega^r = f_j(a^nb^j), \end{aligned}$$

whereas (5) shows that

$$\begin{aligned} (f_{2m-1} * f_j)(a^nb^{2m-1+j}) &= \sum_{r=0}^{2} f_{2m-1}(a^{n-r}b^{2m-1})f_j(a^{-r}b^j) \\ &= \frac{1}{9}\sum_{r=0}^{2} \omega^{n-r}\omega^{-r} = 0. \end{aligned}$$

If now c_m are complex numbers such that $\sum_{-\infty}^{\infty} |c_m| < \infty$, and if

$$(10)\qquad f = f_0 + \sum_{-\infty}^{\infty} c_m f_{2m-1},$$

then

$$\|f\| = 1 + \sum_{-\infty}^{\infty} |c_m| < \infty, \tag{11}$$

and the equations (9) show that $f*f=f$.

Taking infinitely many $c_m \neq 0$, we thus obtain *idempotents on G with infinite support* (and, a fortiori, with infinite support group). The example $f=f_0+f_1$ shows that *there exist idempotents on G with finite support but infinite support group.* Equation (11) shows that *every number* $\geqq 1$ *is the norm of some idempotent on G.* Unless all c_m are 0, *the idempotents* (10) *are not self-adjoint.*

III. I have not succeeded in proving the existence of self-adjoint idempotents with infinite support, but it is easy to give examples in which the support group is infinite.

Put

$$\begin{aligned} a &= (\alpha\beta\gamma)(12)(34)(56)\cdots, \\ b &= (\alpha\beta\gamma)(23)(45)(67)\cdots. \end{aligned} \tag{12}$$

Then ab has infinite order, so that the group G generated by a and b is infinite. The relations $a^2=b^2$, $a^6=b^6=1$ hold. Define $g_1(a^n)=\frac{1}{6}$, $g_1=0$ elsewhere; $g_2(b^n)=\frac{1}{6}\exp\{n\pi i/3\}$, $g_2=0$ elsewhere. Then

$$g_1 * g_1 = g_1, \qquad g_2 * g_2 = g_2, \qquad g_1 * g_2 = g_2 * g_1 = 0. \tag{13}$$

Hence $g=g_1+g_2$ is an idempotent on G whose support S is finite. Since $a \in S$ and $b \in S$, G is the support group of g; and since g_1 and g_2 are self-adjoint, so is g.

IV. Even on a *finite* group there can be uncountably many idempotents, both self-adjoint and non-self-adjoint. To see this, let G be the noncyclic group of order 6, with generators a and b. The relations $a^3=b^2=1$, $ba=a^2b$ hold. If p, q, r are complex numbers, subject to

$$p^2 + pq + q^2 = \frac{1}{12} - r^2, \tag{14}$$

and if

$$\begin{aligned} f(1) &= \tfrac{1}{3}, & f(a) &= -\tfrac{1}{6} + ir, & f(a^2) &= -\tfrac{1}{6} - ir, \\ f(b) &= p + q, & f(ab) &= -p, & f(a^2b) &= -q, \end{aligned} \tag{15}$$

explicit computation shows that $f*f=f$. If r is real and $12r^2<1$, then p and q can be taken real in (14), and the resulting idempotents f are self-adjoint. If r is not real, f is not self-adjoint.

V. We conclude with a positive result:

THEOREM. *If f is an idempotent on G and if $\|f\|=1$, then the support of f is a finite subgroup H of G, and*

$$f(xy) = |H| f(x)f(y) \qquad (x, y \in H). \tag{16}$$

Here $|H|$ denotes the number of elements of H. We sketch the proof. Let S be the support of f, let $m = \max |f(x)|$ $(x \in G)$, and let H be the set of all $x \in G$ at which $|f(x)| = m$. Clearly H is finite. For $x \in H$, we have

$$\left| \sum_y f(y)f(y^{-1}x) \right| = m. \tag{17}$$

Since $\|f\|=1$, (17) is only possible if $y^{-1}x \in H$ for every $y \in S$, i.e., if $S^{-1}H \subset H$. Since $H \subset S$, it follows that H is a group, and then that $S=H$. Also, $|f(x)| = |H|^{-1}$ on H. The equation $f(x) = \sum f(y)f(y^{-1}x)$ then forces the arguments of $f(y)f(y^{-1}x)$ to be equal to the argument of $f(x)$, for all x, $y \in H$, and this gives (16).

Since non-negative idempotents have norm 1 or 0, the above theorem characterizes them as well.

Finally, observe that (16) implies that $f(xy) = f(yx)$ for all x, $y \in G$. In other words, *all idempotents of norm* 1 *lie in the center of the group algebra*. It would be interesting to know whether statement (A) of the Introduction is true for all central idempotents.

REFERENCES

1. P. J. Cohen, *On a conjecture of Littlewood and idempotent measures*, Amer. J. Math. **82** (1960), 191–212.

2. Walter Rudin, *Idempotent measures on abelian groups*, Pacific J. Math. **9** (1959), 195–209.

3. ———, *Fourier analysis on groups*, Interscience, New York, 1962.

UNIVERSITY OF WISCONSIN