fies (F) with $\{(A_1, B_1), \cdots, (A_k, B_k)\}$ as a class of order-pairs. The $2k+1$ integers $k^2, \cdots, k^2+2k$ are reversed by ρ, but two of them must fall in the same set A_i. This is a contradiction.

Therefore G is a proper subgroup of S_∞.

HARVARD UNIVERSITY

ON THE ISOMORPHISM PROBLEM FOR BERNOULLI SCHEMES

BY J. R. BLUM[1] AND D. L. HANSON[2]

Communicated by J. L. Doob, November 27, 1962

1. DEFINITION 1. A Bernoulli scheme $(E, \Omega, \mathfrak{F}, P, T)$ is a probability space together with a transformation T, where

(i) $E=\{1, \cdots, n\}$ for some positive integer n, or $E=\{1, 2, \cdots\}$,

(ii) $\Omega=\{\omega=(\cdots, \omega_{-1}, \omega_0, \omega_1, \cdots) \mid \omega_i \in E \text{ for all } i\}$,

(iii) $\mathfrak{F}$ is the smallest σ-algebra containing all sets $A_i^k=\{\omega \mid \omega_i=k\}$,

(iv) $q_k>0$ is defined for $k \in E$ with $\sum_{k\in E} q_k=1$, P is the product measure on $\mathfrak{F}$ defined by $P\{A_i^k\}=q_k$ for all i,

(v) T is the shift transformation defined on Ω, i.e., $T\omega=\omega'$ if and only if $\omega_i'=\omega_{i+1}$ for all i.

We shall sometimes refer to a Bernoulli scheme as a $(q_1, \cdots, q_n)$-scheme or a $(q_1, q_2, \cdots)$-scheme depending upon whether $E=\{1, \cdots, n\}$ or $E=\{1, 2, \cdots\}$.

DEFINITION 2. Two Bernoulli schemes $(E, \Omega, \mathfrak{F}, P, T)$ and $(E', \Omega', \mathfrak{F}', P', T')$ are said to be *isomorphic modulo sets of measure zero* (or simply *isomorphic*) if there exist sets $D \in \mathfrak{F}$, $D' \in \mathfrak{F}'$ and a mapping $\phi: D \to D'$ such that

(i) $TD=D$,

(ii) $\phi: D \to D'$ is one-to-one and onto,

(iii) $\phi(T\omega)=T'(\phi\omega)$ for all $\omega \in D$,

(iv) if $A \subset D$ then $A \in \mathfrak{F}$ if and only if $\phi A \in \mathfrak{F}'$,

(v) if $A \subset D$ and $A \in \mathfrak{F}$ then $P(A)=P'(\phi A)$,

(vi) $P(D)=1$.

DEFINITION 3. The *entropy* of a $(q_1, \cdots, q_n)$-scheme $[(q_1, q_2, \cdots)$-scheme$]$ is given by

[1] Research supported by the National Science Foundation, Grant NSF G-21219.

[2] Research done under the auspices of the United States Atomic Energy Commission.

$$h = -\sum_i q_i \log q_i.$$

For a detailed discussion of entropy see, e.g., Halmos [1]. It is well known that entropy is an invariant with respect to isomorphism, i.e., any two isomorphic Bernoulli schemes have the same entropy. It is not known whether entropy is a complete invariant, that is, whether two Bernoulli schemes with the same entropy are isomorphic.

In this note we state a theorem which gives conditions under which two Bernoulli schemes are isomorphic. This generalizes results due to Meshalkin [2]. Below is a sketch of Meshalkin's work.

2. Consider a $(q_1, \cdots, q_n)$-scheme and a $(p_1, \cdots, p_m)$-scheme. Let p be a positive integer, let $k_1, \cdots, k_m$ be non-negative integers, and let $M = \sum_\alpha k_\alpha p_\alpha$. Meshalkin calls the $(p_1, \cdots, p_m)$-scheme a *(p, M)-factor scheme* of the $(q_1, \cdots, q_n)$-scheme provided there exist disjoint subsets $I_1, \cdots, I_m$ of $\{1, \cdots, n\}$ such that

(i) $i \in I_\alpha$, $j \in I_\alpha$ implies $q_i = q_j$, $\alpha = 1, \cdots, m$,

(ii) $p_\alpha = \sum_{i \in I_\alpha} q_i = p^{k_\alpha} q_j$ for $j \in I_\alpha$, $\alpha = 1, \cdots, m$.

THEOREM (MESHALKIN). *Consider a $(q_1, \cdots, q_n)$-scheme with entropy h. Then*

(i) *any of its (p, M)-factor schemes has entropy $h - M \log p$,*

(ii) *for fixed p and M any two (p, M)-factor schemes are isomorphic.*

COROLLARY. *A $(q_1, \cdots, q_n)$-scheme and a $(p_1, \cdots, p_m)$-scheme are isomorphic provided*

(i) *they have equal entropy, and*

(ii) *there exist a positive integer p and non-negative integers $k_1, \cdots, k_n$, $r_1, \cdots, r_m$ such that for all i and j with $1 \leqq i \leqq n$, $1 \leqq j \leqq m$ the equations $q_i = p^{-k_i}$ and $p_j = p^{-r_j}$ hold.*

3. DEFINITION 4. Let $(\Omega, \mathfrak{F})$ be a measurable space. A *maximal partition* of $(\Omega, \mathfrak{F})$ is a partition of Ω into measurable disjoint sets such that every measurable subset of Ω is the union of sets in the partition.

DEFINITION 5. Let E be as above and let Σ be the σ-algebra of all subsets of E. Let P be a probability measure defined on Σ which assigns positive probability to each nonempty subset of E. Let $\Sigma_0 \subset \Sigma_1 \subset \Sigma$ be σ-algebras, let $\Pi = (p_1, p_2, \cdots)$ be a finite or infinite sequence of positive numbers with $\Sigma p_i = 1$, and let $0 < \alpha \leqq 1$. Σ_1 is a *simple decomposition of Σ_0 of weight α with respect to* Π if there exist $A_1, A_2, \cdots; B_1, B_2, \cdots; C_1, C_2, \cdots$ all subsets of E such that

(i) $\bigcup_i B_i = \bigcup_i C_i = B$ and $P(B) = \alpha$,

(ii) $\{A_1, A_2, \cdots; B_1, B_2, \cdots\}$ is a maximal partition of (E, Σ_0),

(iii) $\{A_1, A_2, \cdots; B_1C_1, B_1C_2, \cdots; B_2C_1, B_2C_2, \cdots; \cdots\}$ is a maximal partition of (E, Σ_1),

(iv) $P\{B_iC_j\}/P(B_i) = p_j$ for all i and j.

We shall refer to B as the *base* of the decomposition, and to C_i as the *ith compartment* of the decomposition.

DEFINITION 6. *Σ_1 is a decomposition of Σ_0 of weight α with respect to* Π if there exists a finite or infinite sequence of triples $\{(\Sigma^i, B^i, \beta^i)\}$, $i=1, 2, \cdots$, with Σ^i a σ-algebra of subsets of E, B^i a subset of E, and β^i a positive number for each i such that

(i) Σ^i is a simple decomposition of Σ^{i-1} of weight β^i and base B^i with respect to Π $(\Sigma^0 = \Sigma_0)$,

(ii) $B^i \subset B^{i-1}$ for $i \geqq 2$,

(iii) $\sum_i \beta^i = \alpha < \infty$,

(iv) Σ_1 is the smallest σ-algebra containing each Σ^i.

Let D be a finite or denumerably infinite well ordered set (ordered by $\ll$) with initial element i_1.

DEFINITION 7. Σ is a $[D, \{\Pi_i, \alpha_i\}]$ *decomposition of* Σ_0 if for each $i \in D$ there exist sub-σ-algebras $\mathfrak{A}_i$ and $\mathfrak{B}_i$ of Σ such that

(i) $\mathfrak{A}_i$ is a decomposition of $\mathfrak{B}_i$ of weight α_i with respect to Π_i,

(ii) $\mathfrak{B}_i$ is the smallest σ-algebra containing each $\mathfrak{A}_j$ and $j \ll i$,

(iii) Σ is the smallest σ-algebra containing each $\mathfrak{A}_i$,

(iv) $\Sigma_0 = \mathfrak{B}_{i_1}$,

(v) each $e \in E$ is in only a finite number of compartments of simple decompositions.

Now let $(E, \Omega, \mathfrak{F}, P, T)$ and $(E', \Omega', \mathfrak{F}', P', T')$ be Bernoulli schemes. Then P and P' may be considered as probability measures on the σ-algebras Σ and Σ' consisting of all subsets of E and E' respectively. Let $\Sigma_0 = \{\varnothing, E\}$ and $\Sigma_0' = \{\varnothing, E'\}$, where $\varnothing$ is the empty set.

THEOREM. *If there exists a well ordered set D and a sequence $\{\Pi_i, \alpha_i\}$ such that Σ and Σ' are $[D, \{\Pi_i, \alpha_i\}]$ decompositions of Σ_0 and Σ_0' respectively then the two Bernoulli schemes are isomorphic.*

The proof of the theorem will be given elsewhere, together with some applications of the theorem.

REFERENCES

1. P. R. Halmos, *Entropy in ergodic theory*, mimeographed notes, University of Chicago, Chicago, Ill., 1959.

2. L. D. Meshalkin, *A case of isomorphism of Bernoulli schemes*, Dokl. Akad. Nauk SSSR **128** (1959), 31–44. (Russian)

UNIVERSITY OF NEW MEXICO AND
SANDIA CORPORATION