EXAMPLE OF A PROPER SUBGROUP OF S_∞ WHICH HAS A SET-TRANSITIVITY PROPERTY[1]

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S. M. Ulam, on page 33 of his book, *A collection of mathematical problems*, poses the following question: Let G be a subgroup of S_∞ [the group of all permutations of the integers] with the property that for every two sets of integers of the same power whose complements are also of the same power, there exists a permutation g of G which transforms one set into the other. Is $G=S_\infty$ (Chevalley, von Neumann, et al.)?

The answer to this question is no!

We change the problem immaterially by taking S_∞ to be the group of all permutations of the natural numbers rather than the integers; this is helpful since all infinite subsets of the natural numbers are order-isomorphic. A subgroup G which is transitive (wherever possible) on the set of all subsets of the natural numbers is defined by means of a finiteness condition.

Let N be the set of natural numbers with the usual ordering. Consider the set G of all $\sigma \in S_\infty$ satisfying the condition:

(F) there exist $A_1, A_2, \cdots, A_k, B_1, B_2, \cdots, B_k$ subsets of N such that $\bigcup_{i=1}^k A_i = N = \bigcup_{i=1}^k B_i$ and in addition, for all i, $\sigma: A_i \to B_i$ is an order-isomorphism.

Call $\{(A_1, B_1), (A_2, B_2), \cdots, (A_k, B_k)\}$ a class of order-pairs for σ.

Let $\sigma, \tau \in G$ where $\{(A_1, B_1), \cdots, (A_k, B_k)\}$ & $\{(C_1, D_1), \cdots, (C_q, D_q)\}$ are classes of order-pairs for σ & τ respectively. It is easily seen that

$$\{(\sigma^{-1}[B_i \cap C_j], \tau[B_i \cap C_j]) : i \in \{1, \cdots, k\} \;\&\; j \in \{1, \cdots, q\}\}$$

is a class of order-pairs for $\tau\sigma$, so that $\tau\sigma \in G$. Also $\sigma^{-1} \in G$ since $\{(B_1, A_1), (B_2, A_2), \cdots, (B_k, A_k)\}$ is a class of order-pairs for σ^{-1}. Consequently (since G is obviously nonempty) G is a subgroup of S_∞.

That G has the property stated in the problem is clear since subsets of N having the same power are order-isomorphic. The (at most) two order-isomorphisms needed allow us to define an element of G as required.

An element ρ of S_∞ which reverses arbitrarily long strings of natural numbers cannot be in G. For example, ρ can be given by: $\rho(m) = (n+1)^2 - (m+1-n^2)$ where $n^2 \leqq m < (n+1)^2$. Suppose that ρ satis-

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fies (F) with $\{(A_1, B_1), \cdots, (A_k, B_k)\}$ as a class of order-pairs. The $2k+1$ integers $k^2, \cdots, k^2+2k$ are reversed by ρ, but two of them must fall in the same set A_i. This is a contradiction.

Therefore G is a proper subgroup of S_∞.

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ON THE ISOMORPHISM PROBLEM FOR BERNOULLI SCHEMES

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1. DEFINITION 1. A Bernoulli scheme $(E, \Omega, \mathfrak{F}, P, T)$ is a probability space together with a transformation T, where

(i) $E=\{1, \cdots, n\}$ for some positive integer n, or $E=\{1, 2, \cdots\}$,

(ii) $\Omega=\{\omega=(\cdots, \omega_{-1}, \omega_0, \omega_1, \cdots) \mid \omega_i \in E \text{ for all } i\}$,

(iii) $\mathfrak{F}$ is the smallest σ-algebra containing all sets $A_i^k=\{\omega \mid \omega_i=k\}$,

(iv) $q_k>0$ is defined for $k \in E$ with $\sum_{k\in E} q_k=1$, P is the product measure on $\mathfrak{F}$ defined by $P\{A_i^k\}=q_k$ for all i,

(v) T is the shift transformation defined on Ω, i.e., $T\omega=\omega'$ if and only if $\omega_i'=\omega_{i+1}$ for all i.

We shall sometimes refer to a Bernoulli scheme as a $(q_1, \cdots, q_n)$-scheme or a $(q_1, q_2, \cdots)$-scheme depending upon whether $E=\{1, \cdots, n\}$ or $E=\{1, 2, \cdots\}$.

DEFINITION 2. Two Bernoulli schemes $(E, \Omega, \mathfrak{F}, P, T)$ and $(E', \Omega', \mathfrak{F}', P', T')$ are said to be *isomorphic modulo sets of measure zero* (or simply *isomorphic*) if there exist sets $D \in \mathfrak{F}$, $D' \in \mathfrak{F}'$ and a mapping $\phi: D \to D'$ such that

(i) $TD=D$,

(ii) $\phi: D \to D'$ is one-to-one and onto,

(iii) $\phi(T\omega)=T'(\phi\omega)$ for all $\omega \in D$,

(iv) if $A \subset D$ then $A \in \mathfrak{F}$ if and only if $\phi A \in \mathfrak{F}'$,

(v) if $A \subset D$ and $A \in \mathfrak{F}$ then $P(A)=P'(\phi A)$,

(vi) $P(D)=1$.

DEFINITION 3. The *entropy* of a $(q_1, \cdots, q_n)$-scheme $[(q_1, q_2, \cdots)$-scheme$]$ is given by

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