INVARIANT EIGENDISTRIBUTIONS ON SEMISIMPLE LIE GROUPS

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1. Let M be an oriented separable differentiable manifold of dimension n. (We do not assume that M is connected.) Let $C_c^\infty(M)$ denote the space of all complex-valued C^∞ functions on M with compact support. A distribution T on M is a linear mapping $T: C_c^\infty(M)\to\mathbf{C}$ which is continuous in the topology of Schwartz. More explicitly, this means the following. Let U be any open and relatively compact set in M. Then we can select differential operators[2] $D_1, \cdots, D_r$ on M such that

$$|T(f)| \leqq \sum_i \sup |D_i f| \qquad (f\in C_c^\infty(U)).$$

Let G be a group acting on M. We denote by x^g the transform of $x\in M$ by $g\in G$. We assume that, for a fixed g, the mapping $x\to x^g$ of M is of class C^∞. Then for any $f\in C_c^\infty(M)$, the function $f^g: x\to f(x^{g^{-1}})$ is again in $C_c^\infty(M)$ and if T is a distribution, the mapping T^g: $f\to T(f^{g^{-1}})$ $(f\in C_c^\infty(M))$ is also a distribution. We say T is invariant (under G) if $T^g=T$ for all $g\in G$.

Now G operates in a natural way on the spaces[2] of differential operators and differential forms on M. For example if D is a differential operator, $D^gf=(Df^{g^{-1}})^g$ $(f\in C_c^\infty(M),\ g\in G)$. Fix a (real) differential form ω on M of degree n which is invariant under G and which is everywhere positive (with respect to the given orientation of M). Then for every differential operator D on M, we define its adjoint D^* to be the (unique) differential operator satisfying the relation

$$\int_M Df\cdot\phi\omega = \int_M fD^*\phi\cdot\omega$$

for all f, $\phi\in C_c^\infty(M)$. If T is a distribution, the mapping $f\to T(D^*f)$ $(f\in C_c^\infty(M))$ is also a distribution which we denote by DT. Now ω defines a positive Borel measure μ on M. For example if U is an open set in M,

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[2] All differential operators and differential forms are meant to be C^∞ unless explicitly mentioned otherwise.

$$\mu(U) = \int_U \omega.$$

Let F be a function on M which is locally summable (with respect to μ). Then corresponding to F, we get a distribution

$$T_F: f \to \int fFd\mu = \int_M fF\cdot\omega \qquad (f \in C_c^\infty(M)).$$

If T is a distribution, we say $T=F$ if $T=T_F$.

2. Let G be a connected semisimple Lie group. Take $M=G$, $x^g=gxg^{-1}$ ($x, g\in G$) and ω the invariant differential form corresponding to the Haar measure dx on G. Let $\mathfrak{Z}$ be the algebra of all differential operators on G which are invariant under both left and right translations of G. Then $\mathfrak{Z}$ is abelian. Let $l=\text{rank } G$. t being an indeterminate, we denote by $D(x)$ the coefficient of t^l in $\det(t+1-\text{Ad}(x))$ ($x\in G$). Then D is an analytic function on G and an element $x\in G$ is called regular if $D(x)\neq 0$. Let G' be the set of all regular elements in G. Then G' is an open and dense subset of G whose complement is of measure zero.

Let Θ be a distribution on G. We say that it is invariant if $\Theta^x=\Theta$ ($x\in G$) and that it is an eigendistribution of $\mathfrak{Z}$ if $z\Theta=\chi(z)\Theta$ ($z\in\mathfrak{Z}$) for some homomorphism χ of $\mathfrak{Z}$ into $\boldsymbol{C}$.

THEOREM 1. *Let Θ be an invariant eigendistribution of $\mathfrak{Z}$ on G. Then Θ is a locally summable function which is analytic on G'.*

This answers, in particular, a question raised in [**3**, p. 396].

3. Now assume that the center of G is finite. Fix a maximal compact subgroup K of G and let $\mathcal{E}_K$ denote the set of all equivalence classes of irreducible finite-dimensional representations of K. For any $\mathfrak{b}\in\mathcal{E}_K$, let $\xi_\mathfrak{b}$ be the character of $\mathfrak{b}$ and $\mathfrak{b}^*$ the class contragradient to $\mathfrak{b}$ so that[3] $\xi_{\mathfrak{b}^*}(k)=\text{conj } \xi_\mathfrak{b}(k)$ ($k\in K$). For any $f\in C_c^\infty(G)$, define

$$f_\mathfrak{b}(x) = d(\mathfrak{b})\int_K \xi_\mathfrak{b}(k)f(kx)dk \qquad (x \in G),$$

where $d(\mathfrak{b})$ is the degree of any representation in the class $\mathfrak{b}$ and dk is the normalized Haar measure of K. Then $f_\mathfrak{b}\in C_c^\infty(G)$ and the series $\sum_{\mathfrak{b}\in\mathcal{E}_K} f_\mathfrak{b}$ converges in $C_c^\infty(G)$ to f. If T is any distribution on G, the mapping $f\to T(f_{\mathfrak{b}^*})$ ($f\in C_c^\infty(G)$) is also a distribution, which we denote by $T_\mathfrak{b}$. Since

[3] conj c stands for the complex conjugate for $c\in\boldsymbol{C}$.

$$T(f) = \sum_{\mathfrak{d}\in\mathcal{E}_K} T_{\mathfrak{d}}(f) \qquad (f \in C_c^\infty(G)),$$

it is clear that $T_{\mathfrak{d}} \neq 0$ for some $\mathfrak{d} \in \mathcal{E}_K$, if $T \neq 0$.

Now suppose T is an eigendistribution of $\mathfrak{Z}$ on G. Then the same holds for $T_{\mathfrak{d}}$ ($\mathfrak{d} \in \mathcal{E}_K$). But since $T_{\mathfrak{d}}$ transforms, under left translations by elements of K, according to $\mathfrak{d}$, it follows easily that it satisfies an elliptic differential equation on G with analytic coefficients. Therefore $T_{\mathfrak{d}}$ is an analytic function.

4. Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{g}_c$ its complexification. Let G_c be the simply connected complex-analytic group corresponding to $\mathfrak{g}_c$. Assume that G is the real analytic subgroup of G_c corresponding to $\mathfrak{g}$ and rank G = rank K. Fix a maximal connected abelian subgroup A of K and let $\mathfrak{a}$ denote its Lie algebra. Then A is a Cartan subgroup of G and $A' = A \cap G'$ is open and dense in A. Let $\mathfrak{a}_c$ denote the complexification of $\mathfrak{a}$, P the set of all positive roots (under some fixed order) and W the Weyl group of $(\mathfrak{g}_c, \mathfrak{a}_c)$. Then there exists an analytic function Δ on A such that

$$\Delta(\exp H) = \prod_{\alpha\in P} (e^{\alpha(H)/2} - e^{-\alpha(H)/2}) \qquad (H \in \mathfrak{a}).$$

Let $\hat{A}$ denote the character group of A. For any $\hat{a} \in \hat{A}$, define

$$\sigma(\hat{a}) = \prod_{\alpha\in P} \langle \alpha, \lambda \rangle$$

where λ is the linear function on $\mathfrak{a}_c$ such that $\hat{a}(\exp H) = e^{\lambda(H)} (H \in \mathfrak{a})$ and $\langle \alpha, \lambda \rangle$ denotes the usual scalar product defined under the Killing form of $\mathfrak{g}_c$. W operates on $\hat{A}$ in a natural way by duality. An element $\hat{a} \in \hat{A}$ is called regular if its transforms $\hat{a}^s$ ($s \in W$) are all distinct. Then $\hat{a}$ is singular or regular according as $\sigma(\hat{a}) = 0$ or not. Moreover $\sigma(\hat{a}^s) = \epsilon(s)\sigma(\hat{a})$ ($s \in W$, $\hat{a} \in \hat{A}$), where $\epsilon(s) = 1$ or -1 and is independent of $\hat{a}$.

If Θ is an invariant eigendistribution of $\mathfrak{Z}$ on G, one can, in view of Theorem 1, speak of the value $\Theta(x)$ of Θ at any point $x \in G'$. Define the function D as in §2.

THEOREM 2. *Fix a regular element $\hat{a} \in \hat{A}$. Then there exists exactly one invariant eigendistribution $\Theta_{\hat{a}}$ of $\mathfrak{Z}$ on G such that:*

(1) *The function $|D|^{1/2}\Theta_{\hat{a}}$ remains bounded on G';*

(2) *$\Theta_{\hat{a}} = (-1)^q \sigma(\hat{a})\Delta^{-1} \sum_{s\in W} \epsilon(s)\hat{a}^s$ pointwise on A'.*

Here $q = \frac{1}{2}(\dim G - \dim K)$.

For $f, g \in C_c^\infty(G)$, let $f * g$ denote their convolution product so that

$$(f * g)(x) = \int_G f(y)g(y^{-1}x)dy \qquad (x \in G).$$

Also let $\bar{f}(x) = \text{conj}(f(x^{-1}))$.

THEOREM 3. *Put $\Theta = \Theta_{\hat{a}}$ for a fixed regular element $\hat{a}$ in $\hat{A}$. Then $\Theta(\bar{f} * f) \geqq 0$ for every $f \in C_c^\infty(G)$. Moreover the analytic functions $\Theta_{\mathfrak{b}}$ $(\mathfrak{b} \in \mathcal{E}_K)$ all lie in $L_2(G)$.*

It is obvious from its definition that $\Theta \neq 0$. Hence we can choose $\mathfrak{b} \in \mathcal{E}_K$ such that $\Theta_{\mathfrak{b}} \neq 0$. Let V be the smallest closed subspace of $L_2(G)$ containing $\Theta_{\mathfrak{b}}$, which is invariant under the left-regular representation λ of G. Then $V \neq \{0\}$ and it is easy to show that V is the orthogonal sum of a finite number of subspaces which are all invariant and irreducible under λ. This shows that each of the corresponding irreducible representations belongs to the discrete series.

Define $\Theta_{\hat{a}} = 0$ if $\hat{a}$ is a singular element of $\hat{A}$ and let $\mathfrak{H}$ be the smallest closed subspace of $L_2(G)$ which contains every C^∞ eigenfunction of $\mathfrak{Z}$ lying in $L_2(G)$. For any $f \in C_c^\infty(G)$ and $x \in G$, let f_x denote the function $y \rightarrow f(yx)$ $(y \in G)$.

THEOREM 4. *The series*

$$\sum_{\hat{a} \in \hat{A}} \Theta^\wedge(f) \qquad (f \in C_c^\infty(G))$$

converges absolutely and the function

$$f^\#: x \rightarrow \sum_{\in \hat{A}} \theta_{\hat{a}}(f_x) \qquad (x \in G)$$

lies in $\mathfrak{H}$. Moreover the Haar measure of G can be so normalized that $f - f^\#$ is orthogonal to $\mathfrak{H}$ for every $f \in C_c^\infty(G)$.

Theorem 4 shows that our method gives the entire discrete series.

5. The proofs of these results are rather long. We shall only give a brief outline of the main steps in the proofs of Theorems 1 and 2. As before, let $\mathfrak{g}_c$ be the complexification of the Lie algebra $\mathfrak{g}$ of G and $S(\mathfrak{g}_c)$ the symmetric algebra over $\mathfrak{g}_c$. G operates on $\mathfrak{g}_c$ by means of the adjoint representation. Let $I(\mathfrak{g}_c)$ be the subalgebra of all invariants of G in $S(\mathfrak{g}_c)$. Now we take (in the set up of §1) $M = \mathfrak{g}$ and ω the differential form corresponding to the Euclidean measure dX on $\mathfrak{g}$. For $p \in S(\mathfrak{g}_c)$, define the differential operator $\partial(p)$ on $\mathfrak{g}$ as in [4, §2] and identify $\mathfrak{g}_c$ with its dual under the Killing form Ω given by $\Omega(X) = \text{tr}(\text{ad } X)^2 (X \in \mathfrak{g}_c)$. Let $\mathfrak{g}'$ be the set of all regular elements of $\mathfrak{g}$. Then $\mathfrak{g}'$ is open and dense in $\mathfrak{g}$ and its complement is of measure zero.

A subset U of $\mathfrak{g}$ is called completely invariant, if it satisfies the following condition. C being any compact subset of U, $\text{Cl}(C^G) \subset U$. Here $C^G = \bigcup_{x \in G} C^x$ and Cl denotes closure. If U is an open and completely invariant subset of $\mathfrak{g}$, we can take $M = U$ in §1.

LEMMA 1. *Let T be a distribution on a completely invariant open subset U of $\mathfrak{g}$ such that:*

(1) $T^x = T(x \in G)$,

(2) There exists an ideal $\mathfrak{U}$ in $I(\mathfrak{g}_c)$ such that $\dim I(\mathfrak{g}_c)/\mathfrak{U} < \infty$ *and $\partial(u)T = 0$ for $u \in \mathfrak{U}$.*

Then T is a locally summable function on U, which is analytic on $U' = U \cap \mathfrak{g}'$.

This is proved by induction on $\dim \mathfrak{g}$. Let $\mathfrak{N}$ be the set of all $X \in \mathfrak{g}$ such that ad X is nilpotent. The most important step in the proof of Lemma 1 is the following result.

LEMMA 2. *Let T be an invariant distribution on $\mathfrak{g}$ such that*[4] Supp T $\subset \mathfrak{N}$ *and $\partial(\Omega)T = 0$. Then $T = 0$.*

The proof of this makes use of a result of Kostant [6, Corollary 3.7 and Lemma 5.1] from which it follows (see [2, 2.3]) that $\mathfrak{N}$ is the union of a finite number of G-orbits.

In order to obtain Theorem 1, we have now to lift the result of Lemma 1 to the group. For this one needs the following fact.

LEMMA 3. *Let D be a polynomial differential operator* [4, §2] *on $\mathfrak{g}$ such that $D^x = D$ $(x \in G)$ and $Dp = 0$ for $p \in I(\mathfrak{g}_c)$. Then $DT = 0$ for every invariant distribution T on $\mathfrak{g}$.*

The proof again proceeds by induction on $\dim \mathfrak{g}$. The crucial part is the following lemma.

LEMMA 4. *Let T be a distribution and D a polynomial differential operator on $\mathfrak{g}$. We assume that:*

(1) $T^x = T$ $(x \in G)$,

(2) $D^x = D$ and $Dp = 0$ $(x \in G, p \in I(\mathfrak{g}_c))$,

(3) Supp $DT \subset \mathfrak{N}$.

Then $DT = 0$.

First one shows that it is sufficient to consider the case when T is tempered. (This requires a result of Borel, according to which, we can always find a discrete subgroup Γ of G such that G/Γ is compact. See Remark (2) at the bottom of p. 582 of [1].) Now we use the

[4] Supp T denotes the support of T.

theory of Fourier transforms. Put $B(X, Y)=\mathrm{tr}(\mathrm{ad}\, X\ \mathrm{ad}\, Y)(X, Y\in\mathfrak{g})$ and define

$$\hat{f}(Y) = \int e^{iB(Y,X)}f(X)dX \qquad (f\in C_c^\infty(\mathfrak{g}),\ Y\in\mathfrak{g}).$$

Then for any tempered distribution τ, its Fourier transform $\hat{\tau}$ is defined by $\hat{\tau}(f)=\tau(\hat{f})$ $(f\in C_c^\infty(\mathfrak{g}))$. Let J be the ideal of $I(\mathfrak{g}_c)$ spanned by all homogeneous elements of degree $\geqq 1$. Then $\mathfrak{N}$ is exactly the set of zeros of J in $\mathfrak{g}$. Let $p_1, \cdots, p_r$ be an ideal basis for J. Then for every j $(1\leqq j\leqq r)$, we can choose an integer $m_j\geqq 0$ such that $p_j^{m_j}DT=0$ around the origin. Since Supp $DT\subset\mathfrak{N}$ and DT is invariant, it follows that $p_j^{m_j}DT=0$. Let $\mathfrak{U}$ be the ideal in $I(\mathfrak{g}_c)$ generated by $p_j^{m_j}$ $(1\leqq j\leqq r)$. Then $\dim I(\mathfrak{g}_c)/\mathfrak{U}<\infty$ and $uDT=0$ for $u\in\mathfrak{U}$. Hence we conclude from Lemma 1 that $(DT)^\frown$ is a locally summable function. Now define $\hat{D}$ as in [4, p. 91]. Then $(DT)^\frown=\hat{D}\hat{T}$ and it is easy to see that $\hat{D}$ also verifies condition (2) of Lemma 4. From this it follows without difficulty that $\hat{D}\sigma=0$ on $\mathfrak{g}'$ for any invariant distribution σ on $\mathfrak{g}$. Hence $\hat{D}\hat{T}=0$ on $\mathfrak{g}'$. But since $\hat{D}\hat{T}$ is a locally summable function, this implies that $\hat{D}\hat{T}=0$ and therefore $DT=0$.

6. Now we come to Theorem 2. So assume that rank $\mathfrak{g}=$ rank $\mathfrak{k}$ where $\mathfrak{k}$ is the Lie algebra of K. Put $\mathfrak{a}'=\mathfrak{a}\cap\mathfrak{g}'$ and $\pi=\prod_{\alpha\in P}\alpha$. Then π is a polynomial function on $\mathfrak{a}_c$.

LEMMA 5. *Fix $H_0\in\mathfrak{a}'$ and let T be a tempered and invariant distribution on $\mathfrak{g}$ such that*

$$\partial(p)T = p(iH_0)T \qquad (p\in I(\mathfrak{g}_c)).$$

Then if[5] *$T(H)=0$ for $H\in\mathfrak{a}'$, we can conclude that $T=0$.*

LEMMA 6. *Fix $H_0\in\mathfrak{a}'$. Then there exists exactly one tempered and invariant distribution T on $\mathfrak{g}$ such that:*

(1) $\partial(p)T=p(iH_0)T \qquad (p\in I(\mathfrak{g}_c))$,

(2) $T(H)=\pi(H)^{-1}\sum_{s\in W}\epsilon(s)e^{iB(H_0,sH)} \qquad (H\in\mathfrak{a}')$.

The uniqueness of T follows from Lemma 5. The existence is proved as follows. Put

$$\tau(f) = \pi(H_0)\sum_{s\in W}\int_G \hat{f}((sH_0)^x)dx \qquad (f\in C_c^\infty(\mathfrak{g})).$$

Then τ is a tempered and invariant distribution and $\partial(p)\tau=p(iH_0)\tau$

[5] In view of Lemma 1, we can speak of the value $T(X)$ of T at any point X in $\mathfrak{g}'$.

for $p \in I(\mathfrak{g}_c)$ (see [5, pp. 225–226]). Moreover it can be shown that τ satisfies condition (2) of Lemma 6 up to a nonzero constant factor.

Theorem 2 is obtained by lifting the result of Lemma 6 to the group.

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CORRECTION TO *ABSTRACT CLASS FORMATIONS*[1]

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Professor Yukiyosi Kawada has kindly pointed out to us that our construction for an abstract class formation $\{E(K)\}$ is wrong. Namely, we defined $E(K)$ to be a direct limit of a family of groups $\{M(K, N)\}$ under a mapping system $\{\eta^K_{N',N}\}$. These maps $\eta^K_{N',N}$ induce on the second cohomology groups homomorphism whose kernel is not in general 0; hence it is in general not true that $H^2(F, E(k)) = Z(\#F)Z$. For details, see Theorem 2 of a paper by Kawada, forthcoming in Boletim da Sociedade de Matemática de São Paolo.

Our main theorem that a class formation does exist for every G_∞, is however true: this is proved by Kawada in the paper just mentioned, using the same family of groups $M(K, N)$ but taking an inverse limit.

After seeing Kawada's work, one of us has found a correct construction using a direct limit and replacing the $\{\eta^K_{N',N}\}$ by a different system of maps. This will be explained in a paper to be published elsewhere.

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[1] Bull. Amer. Math. Soc. **67** (1961), 393–395.