POSITIVE SOLUTIONS OF THE HEAT EQUATION[1]

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It is the purpose of this note to set forth several new integral representations of solutions of the heat equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \tag{1}$$

which are positive for all x and for all negative or for all positive t. These results are consequences of the author's study of the Appell transformation:

$$v(x, t) = k(x, t)u(x/t, -1/t). \tag{2}$$

Here $k(x, t)$ is the fundamental solution of (1),

$$k(x, t) = (4\pi t)^{-1/2}e^{-x^2/4t}.$$

The transformation is known to carry a solution u of (1) into another v, and it serves in a remarkable way to set up a duality between various classes of solutions. Proofs of the following results will appear in the Transactions of the American Mathematical Society.

THEOREM 1. *A necessary and sufficient condition that a function $u(x, t)$ should have the integral representation*

$$u(x, t) = \int_{-\infty}^{\infty} e^{xy+ty^2}d\alpha(y) \tag{3}$$

for $-\infty < t < 0$, with $\alpha(y)$ nondecreasing, is that $u(x, t)$ should satisfy (1) *and be non-negative there.*

An example of such a function is $e^t \cosh x$, with $\alpha(y)$ a step-function. This representation may be used to give an immediate proof of a theorem of I. I. Hirschman [1] concerning solutions of (1) for $t<0$ which turn out to be constant as a result of restricted growth properties, $x \to \pm\infty$, $t = t_0$.

THEOREM 2. *A necessary and sufficient condition that a function $u(x, t)$ should have the representation*

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$$u(x, t) = \int_{-\infty}^{\infty} k(y + ix, -t)\phi(y)dy$$

for $-\infty < t < 0$, *with* $\phi(y)$ *positive definite, is that* $u(x, t)$ *should satisfy* (1) *and be non-negative there and in addition that*

$$\int_{-\infty}^{\infty} u(x, t_0)e^{x^2/4t_0}dx < \infty$$

for some $t_0 < 0$.

An example of such a function is $k(ix, 1-t)$ with $\phi(y)$ equal to the positive definite function $(4\pi)^{-1/2}e^{-y^2/4}$.

THEOREM 3. *A necessary and sufficient condition that a function* $u(x, t)$ *should have the representation*

$$u(x, t) = \int_{-\infty}^{\infty} e^{ixy - ty^2}\phi(y)dy \tag{4}$$

for $0 < t < \infty$, *with* $\phi(y)$ *positive definite, is that* $u(x, t)$ *should satisfy* (1) *and be non-negative there and in addition that*

$$\int_{-\infty}^{\infty} u(x, t_0)dx < \infty \tag{5}$$

for some $t_0 > 0$.

An example here is $k(x, t)$ with $\phi(y)$ equal to the constant $(2\pi)^{-1}$. A positive solution of (1) which fails to have the representation (4) is $x^2 + 2t$. It does not satisfy (5) for any $t_0 > 0$.

REFERENCE

1. I. I. Hirschman, *A note on the heat equation*, Duke Math. J. 19 (1952), 487–492.

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