EXISTENCE THEOREM FOR THE BARGAINING SET $\mathbf{M}_1^{(i)}$

BY BEZALEL PELEG

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M. Davis and M. Maschler have conjectured [1] that for each coalition structure[1] $\mathbf{B}$ in a cooperative game, there exists a payoff vector $\mathbf{x}$ such that the payoff configuration $(\mathbf{x}; \mathbf{B})$ is stable, i.e., belongs to the bargaining set $\mathbf{M}_1^{(i)}$. We outline here a proof of the conjecture.[2] The details of the proof will be published elsewhere.

Let $\mathbf{B} \equiv B_1, B_2, \cdots, B_m$ be a fixed coalition structure for an n-person game Γ with a characteristic function $v(B)$, satisfying $v(B) \geqq 0$, and $v(i) = 0$ for $i = 1, 2, \cdots, n$. We denote by $X(\mathbf{B})$ the space of the points $\mathbf{x} \equiv (x_1, x_2, \cdots, x_n)$ such that $(\mathbf{x}; \mathbf{B})$ is an individually rational payoff configuration (i.r.p.c.). Thus, $X(\mathbf{B}) = S_1 \times S_2 \times \cdots \times S_m$, where for $j = 1, 2, \cdots, m$, S_j is the simplex

$$\left\{\hat{\mathbf{x}}^{B_j} \equiv \{x_k\}_{k\in B_j}: x_k \geqq 0 \text{ and } \sum_{k\in B_j} x_k = v(B_j)\right\}.$$

LEMMA. *Let $c^1(\mathbf{x}), c^2(\mathbf{x}), \cdots, c^n(\mathbf{x})$ be non-negative continuous real functions defined for $\mathbf{x} \in X(\mathbf{B})$. If, for each $\mathbf{x}$ in $X(\mathbf{B})$, and for each coalition B_j in $\mathbf{B}$, there exists a player i in B_j, such that $c^i(\mathbf{x}) \geqq x_i$, then there exists a point $\xi \equiv (\xi_1, \xi_2, \cdots, \xi_n)$ in $X(\mathbf{B})$ such that $c^k(\xi) \geqq \xi_k$ for all k, $k = 1, 2, \cdots, n$.*

The proof is indirect and one arrives at the contradiction by using Brouwer's fixed point theorem.

Let $(\mathbf{x}; \mathbf{B})$ be an i.r.p.c. We shall denote by $(\mathbf{y}^{B_j}, \hat{\mathbf{x}}^{N-B_j}; \mathbf{B})$ an i.r.p.c. which results from the previous one by holding the payments to the players in $N - B_j$ fixed, and giving each player k in B_j, $B_j \in \mathbf{B}$, an amount y_k. Clearly, $\mathbf{x}^{N-B_j}$ is the projection of $\mathbf{x}$ on the space $S_1 \times \cdots \times S_{j-1} \times S_{j+1} \times \cdots \times S_m$, and $\hat{\mathbf{y}}^{B_j} \equiv \{y_k\}$ is a point in S_j.

Let $E_j^i(\mathbf{x})$ be the set of points $\hat{\mathbf{y}}^{B_j}$ in S_j, having the property that in $(\hat{\mathbf{y}}^{B_j}, \hat{\mathbf{x}}^{N-B_j}; \mathfrak{B})$, player i, $i \in B_j$, is *not weaker* than any other player. The set $E_j^i(\mathbf{x})$ is closed and contains the face $y_i = 0$ of S_j. (See [2].)

We now define for each player i, $i = 1, 2, \cdots, n$, the function

$$c^i(\mathbf{x}) \equiv x_i + \max_{\hat{\mathbf{y}}^{B_j} \in E_j^i(\mathbf{x})} \min_{k \in B_j} (x_k - y_k).$$

Here, B_j is that coalition of $\mathbf{B}$ which contains player i.

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[1] Throughout this paper we shall use the definitions and the notations of [2].

[2] Another proof has been given by the author, M. Davis, and M. Maschler. It has been decided to publish this version, which is simpler.

It can be shown that $c^i(\mathfrak{x})$ is a non-negative continuous function of $\mathbf{x}$.

Since $\sum_{k\in B_j} x_k = \sum_{k\in B_j} y_k = v(B_j)$, it follows that $c^i(\mathbf{x}) \leqq x_i$ for all i, $i=1, 2, \cdots, n$. Let E_i, $i=1, 2, \cdots, n$, be the set of points $\mathbf{x}$, $\mathbf{x}\in X(\mathbf{B})$, for which i is *not weaker* than any other player of the coalition B_j which contains player i. Clearly, $(\mathbf{x};\ \mathbf{B})\in \mathbf{M}_1^{(i)}$ if and only if $\mathbf{x}\in \bigcap_{k=1}^n E_k$. If $\mathbf{x}\in E_i$, then its projection $\hat{\mathbf{x}}^{B_j}$ on S_j belongs to $E_j^i(\mathbf{x})$. In this case $c^i(\mathbf{x}) = x_i$. Conversely, if $c^i(\mathbf{x}) = x_i$, then some $\hat{\mathbf{y}}^{B_j}\in E_j^i(\mathbf{x})$ must be equal coordinatewise to $\mathbf{x}^{B_j}$, hence $\mathfrak{x}\in E_i$.

It is proved in [2] (see proof of Theorem 2), that for each $\mathbf{x}$, $\mathbf{x}\in X(\mathbf{B})$, and for each coalition B_j, $B_j\in \mathbf{B}$, there exists a player i, $i\in B_j$, such that $\mathbf{x}\in E_i$. Thus, for this player, $c^i(\mathbf{x}) = x_i$. By the lemma, there exists a point ξ, $\xi\in X(\mathbf{B})$, such that $c^k(\xi) = \xi_k$ for all k, $k=1, 2, \cdots, n$. Therefore, $\xi\in \bigcap_{k=1}^n E_k$, and so $(\xi,\ \mathbf{B})\in \mathbf{M}_1^{(i)}$. We have thus proved:

THEOREM. *Let* $\mathbf{B}$ *be a coalition structure in an n-person cooperative game*; *then there always exists a payoff vector* $\mathbf{x}$ *such that* $(\mathbf{x};\ \mathbf{B})\in \mathbf{M}_1^{(i)}$.

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REFERENCES

1. M. Davis and M. Maschler, *Existence of stable payoff configurations for cooperative games*, Econometric Research Program, Research Memorandum No. 38 (1962), Princeton University, Princeton, N. J.

2. ———, *Existence of stable payoff configurations for cooperative games*, Bull. Amer. Math. Soc. 69 (1963), 106–108.

HEBREW UNIVERSITY OF JERUSALEM