EXISTENCE OF STABLE PAYOFF CONFIGURATIONS FOR COOPERATIVE GAMES

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1. Let Γ be an n-person cooperative game with a (not necessarily superadditive) characteristic function $v(B)$. We assume that Γ is normalized so that $v(B) \geqq 0$ for each coalition B, and $v(i) = 0$ for $i = 1, 2, \cdots, n$. Let $\mathbf{B} \equiv B_1, B_2, \cdots, B_m$ be a coalition structure, i.e., a partition of the set $N = \{1, 2, \cdots, n\}$ into m nonempty coalitions. An outcome of the game with this coalition structure can be represented by a *payoff configuration* $(\mathbf{x}; \mathbf{B})$, where the payoff vector $\mathbf{x} \equiv (x_1, x_2, \cdots, x_n)$ represents the amount which the players receive. If we restrict ourselves to *individually rational* payoff configurations (i.r.p.c.'s), i.e., to payoff configurations with $\mathbf{x} \geqq 0$ coordinatewise, then $\mathbf{x}$ must lie in the space $X(\mathbf{B}) \equiv S_1 \times S_2 \times \cdots \times S_m$, where $S_j \equiv \{\hat{\mathbf{x}}^{B_j} \equiv \{x_k\}_{k \in B_j}: x_k \geqq 0 \text{ and } \sum_{k \in B_j} x_k = v(B_j)\}$ are geometric simplices for $j = 1, \cdots, m$.

Let $(\mathbf{x}; \mathbf{B})$ be an i.r.p.c. for a game Γ, and let ν and μ be two distinct players in a coalition B_j of $\mathbf{B}$. An *objection* of ν against μ in $(\mathbf{x}; \mathbf{B})$ is a vector $\hat{\mathbf{y}}^C$, where C is a coalition containing player ν but not player μ, whose coordinates $\{y_k\}$, $k \in C$, satisfy: $y_\nu > x_\nu$, $y_k \geqq x_k$ and $\sum_{k \in C} y_k = v(C)$. A *counter objection* to this objection is a vector $\hat{\mathbf{z}}^D$, where D is a coalition containing player μ but not player ν, whose coordinates $\{z_k\}$, $k \in D$, satisfy: $z_k \geqq x_k$ for each k in D, $z^k \geqq y^k$ for each k in $C \cap D$, $\sum_{k \in D} z_k = v(D)$.

DEFINITION. We shall say that player ν is *stronger* than player μ (or, equivalently, that player μ is *weaker* than player ν), in $(\mathbf{x}; \mathbf{B})$, if ν has an objection against μ, which cannot be countered. We denote this by $\nu \succ \mu$. We shall say that both players are equal, and write $\nu \sim \mu$, if neither $\nu \succ \mu$ nor $\mu \succ \nu$.

REMARK. By definition, $\nu \sim \mu$ in $(\mathbf{x}; \mathbf{B})$ if ν and μ belong to different coalitions of $\mathbf{B}$.

DEFINITION. A coalition B_j in $\mathbf{B}$ will be called *stable* in $(\mathbf{x}; \mathbf{B})$, if each two of its members are equal.

DEFINITION. An i.r.p.c. $(\mathbf{x}; \mathbf{B})$ is called *stable* if each coalition in $\mathbf{B}$ is stable in $(\mathbf{x}; \mathbf{B})$.

The set of all the stable i.r.p.c.'s is called the *bargaining set* $\mathbf{M}_1^{(i)}$ of

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the game Γ. It was first introduced by R. J. Aumann and M. Maschler [1].

2. Let $(\mathbf{x};\ \mathbf{B})$ be an i.r.p.c. for a game Γ and let C be a coalition. Then $e(C) \equiv v(C) - \sum_{i\in C} x_i$ will be called the *excess* of C in $(\mathbf{x};\ \mathbf{B})$. The following lemma follows from the definitions:

LEMMA 1. *If, in* $(\mathbf{x};\ \mathbf{B})$, *player* ν *has an objection* $\hat{\mathbf{y}}^C$ *against player* μ, *and this objection cannot be countered, then each coalition* D, *for which* $\mu \in D$, $e(D) \geqq e(C)$, *must contain player* ν.

THEOREM 1. *Let* $(\mathbf{x};\ \mathbf{B})$ *be an i.r.p.c. for a game* Γ; *then the partial relation* $\succ$ *in* $(\mathbf{x};\ \mathbf{B})$ *is never intransitive.* (*Also, it is asymmetric.*)

PROOF. Suppose, on the contrary, that a coalition B_j, $B_j \in \mathbf{B}$, contains the players, say, $1, 2, \cdots, t$ and that $1 \succ 2$, $2 \succ 3, \cdots, t-1 \succ t$, $t \succ 1$. Then, in $(\mathbf{x};\ \mathbf{B})$, there exists an objection $\hat{\mathbf{y}}^{C_\nu}$ of player ν against player $(\nu+1) \pmod t$, which cannot be countered, $\nu = 1, 2, \cdots, t$. Let C_{ν_0} be a coalition which has the maximum excess among the C_ν's. It follows by induction, using Lemma 1, that C_{ν_0} contains all the players $1, 2, \cdots, t$. This is impossible, since it cannot contain player $(\nu_0 - 1) \pmod t$.

The 5-person game, where $v(123) = 30$, $v(14) = 40$, $v(35) = 20$, $v(245) = 30$, $v(B) = 0$ otherwise, furnishes an example in which the relation $\succ$ is *not* transitive. Indeed, $1 \succ 2$, $2 \succ 3$, $1 \sim 3$ in $(10, 10, 10, 0, 0;\ 123, 4, 5)$. A similar example can be constructed to show that the relation $\sim$ is not necessarily transitive.

3. Let $(\mathbf{x};\ \mathbf{B})$ be an i.r.p.c. for a game Γ. We shall denote by $(\hat{\mathbf{y}}^{B_j}, \hat{\mathbf{x}}^{N-B_j};\ \mathbf{B})$ an i.r.p.c. which results from the previous one by holding fixed the payments to the players in $N - B_j$, and giving each player k in B_j, $B_j \in \mathbf{B}$, an amount y_k. Clearly, $\hat{\mathbf{y}}^{B_j} = \{y_k\}$, $k \in B_j$, is a point in S_j. Let $E_j^i(\mathbf{x})$ be the set of points $\hat{\mathbf{y}}^{B_j}$ in S_j having the property that in $(\mathbf{y}^{B_j}, \hat{\mathbf{x}}^{N-B_j};\ \mathbf{B})$, player i, $i \in B_j$, is *not weaker* than any other player. The set $E_j^i(\mathbf{x})$ is closed and contains the face $y_i = 0$ of the simplex S_j. (If $y_i = 0$, player i can always counter object by playing as a 1-person coalition.) We shall prove that $M_j(\mathbf{x}) \equiv \bigcap_{i\in B_j} E_j^i(\mathbf{x}) \neq \emptyset$. Indeed, in view of the lemma of B. Knaster, C. Kuratowski, and S. Mazurkiewicz [3], it suffices to prove that $\bigcup_{i\in B_j} E_j^i(\mathbf{x}) = S_j$; i.e., that for any i.r.p.c. $(\mathbf{x};\ \mathbf{B})$, and any coalition B_j in $\mathbf{B}$, there exists a player i, $i \in B$, such that $i \succsim k$ in $(\mathbf{x};\ \mathbf{B})$ for all k. If this is not the case for an i.r.p.c. $(\mathbf{x};\ \mathbf{B})$, one arrives at a contradiction to Theorem 1. We have thus proved:

THEOREM 2. *Let* $(\mathbf{x};\ \mathbf{B})$ *be an i.r.p.c. for a game* Γ, *and let* $B_j \in \mathbf{B}$. *It* *is possible to modify the payments to the players in B_j, without changing the payments to the players in $N-B_j$, in such a way that B_j becomes stable.*

COROLLARY. *There always exists an* $\mathbf{x}$ *such that* $(\mathbf{x}; N) \in \mathbf{M}_1^{(i)}$.

4. On the basis of these results, B. Peleg presents an ingenious proof in the subsequent research announcement [4], that for each coalition structure $\mathbf{B}$, for a game Γ, there exists a payoff vector $\mathbf{x}$, such that $(\mathbf{x}; \mathbf{B}) \in \mathbf{M}_1^{(i)}$. His proof is indirect, and does not furnish more properties of $\mathbf{M}_1^{(i)}$. Therefore, a direct proof is also desirable.

There exist examples which show that the sets $M_j(\mathbf{x})$ are *not* necessarily convex. From the definitions of these sets it follows that they are closed polyhedra. If one could show that these polyhedra are *acyclic*, i.e., have only trivial homology groups, then one could use the Eilenberg-Montgomery fixed-point theorem [2] to prove Peleg's result in a direct fashion. So far, we know that $M_j(\mathbf{x})$ is acyclic if the coalition B_j contains less than 4 players, and we did not find counterexamples for larger coalitions. We also know that $E_j^i(\mathbf{x})$ are always contractible over themselves to a point, and hence they are acyclic. These results follow from:

LEMMA 2. *If* $1 \succ 2$ *in* $(\mathbf{x}; \mathbf{B})$, $1, 2 \in B_j \in \mathbf{B}$, *and if* $\hat{\mathbf{y}}^{B_j}$ *is a point in* S_j *such that* $y_1 \leqq x$, $y_2 \geqq x_2$, *and*

$$x_1 - y_1 \leqq \sum_{i \in P} (y_i - x_i),$$

where P *is the set of players in* B_j, *different from player* 2, *for which* $y_i > x_i$, *then* $1 \succ 2$ *also in* $(\hat{\mathbf{y}}^{B_j}, \mathbf{x}^{N-B_j}; \mathbf{B})$.

The proof is straightforward, once one realizes that these conditions make it "more difficult" for player 2 to object and "easier" for player 1 to counter object.

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