UNKNOTTING S^1 IN S^4

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Topologists have for some time suspected that the k-sphere S^k can be topologically knotted in the n-sphere S^n if and only if $k>0$ and $n-k=2$. Strictly speaking, this is not quite correct (because of the existence of wild embeddings), but with the appropriate local flatness condition, the conjecture has been verified by Brown [1; 2] for $n-k=1$, Artin [3] for $n-k=2$, and Stallings [4] for $n-k\geqq 3$, the single undecided case occurring when $k=1$ and $n=4$.

It is the object of this note to show that, on the basis of some recent results of Homma, S^1 can not be knotted in S^4.

1. **The main theorem.** R^n will denote n-dimensional Euclidean space, and we identify R^n with $R^n\times 0\subset R^{n+1}$ so that we may write $R^n\subset R^{n+1}$. The unit sphere in R^{n+1} will be denoted by S^n. S^n can be triangulated as a combinatorial manifold so that, for each $k<n$, S^k appears as a subcomplex.

Let f be an embedding of a k-manifold M^k in an n-manifold M^n with the property that each point of $f(M^k)$ has a neighborhood U in M^n such that the pair $(U, U\cap f(M^k))$ is homeomorphic to the pair (R^n, R^k). Then f is called a *locally flat* embedding and $f(M^k)$ is called a *locally flat* submanifold of M^n.

The main theorem of this paper will be

THEOREM 1.1. *Let f_1 and f_2 be locally flat embeddings of S^1 in S^4. Then there is a homeomorphism h of S^4 onto itself such that*

$$hf_1 = f_2.$$

Furthermore, if p is a point of $S^4-f_1(S^1)-f_2(S^1)$, then h can be chosen so as to restrict to the identity in some neighborhood of p.

Since a general position argument will prove Theorem 1.1 whenever f_1 and f_2 happen to be piecewise linear embeddings, it will be more than sufficient to prove the following theorem, in which $U_\epsilon(f(S^1))$ denotes the set of points in S^4 whose distance from $f(S^1)$ is less than ϵ.

THEOREM 1.2. *Let f be a locally flat embedding of S^1 in S^4. Then for any $\epsilon>0$, there is an ϵ-homeomorphism h of S^4 onto itself such that*

$$h/S^4 - U_\epsilon(f(S^1)) = 1,$$

$$hf\colon S^1 \to S^4 \text{ is piecewise linear.}$$

2. **Homma's results.** Homma [5] has recently proved the following theorem.

HOMMA'S THEOREM. *Let the following be given:*

(i) M^n, *a finite combinatorial n-manifold;*

(ii) $\tilde{M}^n$, *a finite combinatorial n-manifold topologically embedded in* M^n;

(iii) $\tilde{P}^k$, *a finite polyhedron piecewise linearly embedded in* $\operatorname{int}(\tilde{M}^n)$. *If* $2k+2\leqq n$, *then for any* $\epsilon>0$ *there is an* ϵ*-homeomorphism* F *of* M^n *onto* M^n *such that*

$$F/M^n - U_\epsilon(\tilde{P}^k) = 1,$$
$$F/\tilde{P}^k \textit{ is piecewise linear.}$$

With only slight modifications, Homma's arguments are sufficient to produce the following somewhat more general result.

THEOREM 2.1. *Let the following be given:*

(i) M^n, *a possibly noncompact combinatorial n-manifold;*

(ii) $\tilde{M}^n$, *a possibly noncompact combinatorial n-manifold, topologically embedded in* M^n;

(iii) $\tilde{P}^k$, *a possibly infinite polyhedron, piecewise linearly embedded as a closed subset of* $\operatorname{int}(\tilde{M}^n)$;

(iv) $\tilde{L}$, *a subpolyhedron of* $\tilde{P}^k$ *such that* $\operatorname{Cl}(\tilde{P}^k-\tilde{L})$ *is a finite polyhedron, and such that* $\tilde{L}$ *is piecewise linearly embedded in* M^n *as well as in* $\tilde{M}^n$.

If $2k+2\leqq n$, *then for any* $\epsilon>0$ *there is an* ϵ*-homeomorphism* F *of* M^n *onto* M^n *such that*

$$F/M^n - U_\epsilon(\tilde{P}^k - \tilde{L}) = 1,$$
$$F/\tilde{L} = 1,$$
$$F/\tilde{P}^k \textit{ is piecewise linear.}$$

3. **Proof of the main theorem**

LEMMA 3.1. *Let* α *be an open arc in* S^4, *and* u,v,w,x *four points on* α, *in that order. Let* U *and* V *be open neighborhoods in* S^4 *of the closed subarcs* $[uw]$ *and* $[vx]$, *respectively, of* α, *such that* $(U, U\cap\alpha)\approx(R^4, R^1)\approx(V, V\cap\alpha)$. *Then there is an open neighborhood* W *of* $[ux]$ *in* S^4 *such that* $(W, W\cap\alpha)\approx(R^4, R^1)$.

Since $(U, U\cap\alpha)\approx(R^4, R^1)$, there is a homeomorphism h of U onto itself which takes $U\cap\alpha$ onto itself, u onto v and w onto itself, and is the identity near the boundary of U. Extend h over S^4 via the identity, and let $W=h^{-1}(V)$.

Repeated use of this lemma proves the following

THEOREM 3.2. *Let S be a locally flat 1-sphere in S^4. Then S may be written as the union of two open arcs, A and B, which have neighborhoods, U_A and U_B, in S^4 such that*

(i) *$U_A \cap S = A$ and $(U_A, A) \approx (R^4, R^1)$;*

(ii) *$U_B \cap S = B$ and $(U_B, B) \approx (R^4, R^1)$.*

Now let f be a locally flat embedding of S^1 in S^4, and $\epsilon > 0$ a given positive number. Theorem 1.2 will be proved by a double application of Homma's theorem, first in its original form and then in the form of Theorem 2.1.

PROOF OF THEOREM 1.2. Write $f(S^1)$ as the union of two open arcs A and B as in the above theorem, and let x and y be two points of $f(S^1)$, one chosen from each of the two components of $A \cap B$. Then $f(S^1)$ is the union of the two closed arcs $a \subset A$ and $b \subset B$, which intersect at x and y.

Step 1. Since $(U_A, A) \approx (R^4, R^1)$, U_A can be triangulated as a combinatorial manifold in such a way as to make

$$f: f^{-1}(a) \to a \subset U_A$$

a piecewise linear embedding.

Let $M^n = S^4$, $\tilde{M}^n =$ a closed regular neighborhood of a in U_A, and $\tilde{P}^k = a$. Homma's theorem then asserts the existence of an $\epsilon/2$-homeomorphism F_1 of S^4 onto itself such that

$$F_1/S^4 - U_{\epsilon/2}(a) = 1,$$

$$F_1/a \text{ is piecewise linear.}$$

Step 2. Since $(F_1(U_B), F_1(B)) \approx (U_B, B) \approx (R^4, R^1)$, $F_1(U_B)$ can be triangulated as a combinatorial manifold in such a way as to make

$$F_1 f: f^{-1}(B) \to F_1(B) \subset F_1(U_B)$$

a piecewise linear embedding.

For the second application of Homma's theorem, let $M^n = F_1(U_B)$ triangulated as an open subset of S^4, $\tilde{M}^n = F_1(U_B)$ triangulated as in the preceding paragraph, $\tilde{P}^k = F_1(B)$ and $\tilde{L} = F_1(B) \cap F_1(a)$. Note that by choice of F_1, $\tilde{L}$ is piecewise linearly embedded in M^n as well as in $\tilde{M}^n$. Now apply Theorem 2.1 to obtain an $\epsilon/2$-homeomorphism F_2 of $F_1(U_B)$ onto itself such that

$$F_2/F_1(U_B) - U_{\epsilon/2}(F_1(B) - F_1(a)) = 1,$$

$$F_2/F_1(B) \cap F_1(a) = 1,$$

$$F_2/F_1(B) \text{ is piecewise linear.}$$

F_2, which is the identity near the boundary of $F_1(U_B)$, may be extended via the identity to a homeomorphism F_2 of S^4 onto itself.

Then $h = F_2F_1$ is an ϵ-homeomorphism of S^4 onto itself such that

$$h/S^4 - U_\epsilon(f(S^1)) = 1,$$

$$hf: S^1 \to S^4 \text{ is piecewise linear.}$$

This completes the proof of Theorem 1.2, and hence also of Theorem 1.1.

Theorem 1.2 is actually a very special case of a more general result which will be described elsewhere.

References

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