# FREE AND DIRECT OBJECTS

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1. **General considerations.** Let $\mathcal{B}$ be a bicategory;[2] the following terms are supposed to be familiar to the reader: object, morphism (=map of the class in the question), equivalence (=isomorphism) injection, surjection (=projection in the sense of [**13**; **9**]). A morphism $\alpha: A \to B$ is called a *retraction* (and $B$ is called a *retract* of $A$) if there exists a cross-section $\beta: B \to A$ i.e., a morphism such that $\alpha\beta$ is the identity $\epsilon_B: B \to B$. If this is the case, $\alpha$ must be a surjection and $\beta$ must be an injection. Map $(A, B)$ will denote the set of all morphisms $\alpha: A \to B$.

An object $S$ will be called a *singleton* if Map$(S, A)$ is not void and Map$(A, S)$ consists of exactly one morphism for every object $A$; dually $S$ is a *cosingleton* if Map$(A, S) \neq \emptyset$ and Map$(S, A)$ consists of exactly one morphism for every $A$. All singletons and cosingletons are equivalent (if they exist). $S$ is a singleton and a cosingleton simultaneously if and only if it is a null object. An example of a singleton which is not a null object is a one-point space in the category of topological spaces.

$\{A_t\}_{t \in T}$ being a set of objects, $\Sigma A_t$ and $\Pi A_t$ will denote the free and direct join of it (cf. [**12**, §12]) with monomorphisms $\sigma_t: A_t \to \Sigma A_u$ and epimorphisms $\pi_t: \Pi A_u \to A_t$, respectively.

PROPOSITION 1. *If $\mathcal{B}$ has a singleton or a cosingleton, then the monomorphisms $\sigma_t: A_t \to \Sigma A_u$ are injections admitting retractions $\pi_t: \Sigma A_u \to A_t$ and, dually, the epimorphisms $\pi_t: \Pi A_u \to A_t$ are surjections admitting cross-sections $\sigma_t: A_t \to \Pi A_u$.*

According to the standard definition an object $P$ is *projective* if for every surjection $\alpha: A \to B$ and every $\beta: P \to B$ there exists $\gamma: P \to A$ such that $\alpha\gamma = \beta$, and $I$ is *injective* if for every injection $\alpha: B \to A$ and every $\beta: B \to I$ there exists $\gamma: A \to I$ with $\gamma\alpha = \beta$.

PROPOSITION 2. *The retracts and free joins of projective objects are projective; the retracts and direct joins of injective objects are injective.*

An object $M$ with be called a *coseparator* if for any two objects $A$ and $B$ and for any morphisms $\alpha: A \to B$ and $\beta: A \to B$, the condition $\alpha\gamma = \beta\gamma$ for all $\gamma \in$ Map$(M, A)$ implies $\alpha = \beta$. Let us notice that any

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[2] We assume Isbell's system of axioms, cf. [9], also [5; 7; 12; 13].

coseparator is a generator in the sense of [**7**].

An object $F$ will be called a *basic free object* (abbreviation: b.f.o.) if the following conditions are satisfied: (1) $F$ is a coseparator. (2) If a coseparator $A$ is a retract of $F$, then $A$ is equivalent to $F$. (3) If $B$ is any coseparator, then there exists a retraction $\alpha: B \to F$. A b.f.o. $F$ will be called *strict* if the following condition holds: If $\alpha: B \to A$ is an injection and the conjugate map $\alpha'$: $\text{Map}(F, B) \to \text{Map}(F, A)$ is onto, then $\alpha$ is an equivalence.

$F$ is unique up to equivalence (if it exists). An object $P$ will be called *free* if it is a free join of a set of copies of $F$. Dually we define: a *separator*, a *basic direct object* $D$ (abbreviation: b.d.o.), a *direct* object, a *strict* b.d.o.

THEOREM. *Suppose that* $\mathcal{B}$ *admits a singleton or a cosingleton and free [direct] objects and that b.f.o. is strict and projective [b.d.o. is strict and injective]. Then*

(i) *Every free [direct] object is projective [injective].*

(ii) *Every object is an image of a free object [a subobject of a direct object] (i.e., for each* $A$ *there exists a surjection* $\alpha: \Sigma(F)_t \to A$ *[an injection* $\alpha: A \to \Pi(D)_t$*]).*

(iii) *An object is projective [injective] if and only if it is a retract of a free [direct] object.*

(iv) *An object* $A$ *is projective [injective] if and only if for every object* $B$ *every surjection* $\alpha: B \to A$ *has a cross-section [every injection* $\alpha: A \to B$ *has a retraction]. In other words,* $A$ *is projective [injective] if and only if it is an absolute quotient retract [absolute subretract].*

2. **Examples.** 1. If $\mathcal{B}$ is the bicategory of abelian groups and homomorphisms, then the infinite cyclic group $Z$ is a b.f.o. and $Q/Z$ is a b.d.o. where $Q$ denotes the group of rationals. Free objects are just free abelian groups, every projective is free and injective objects are just the divisible groups (cf. [**11**, §12; **13**]).

2. If $\mathcal{B}$ is the bicategory of all groups and homomorphisms then $Z$ is a b.f.o. Free objects are the same as free groups and projective objects are the same as free (Nielsen-Schreier theorem, cf. [**11**, §35]). The only injective object is the null object [**3**, Theorem 2].

3. If $\mathcal{B}$ is the bicategory of Boolean algebras and Boolean homomorphisms, then the two-element algebra (0,1) is a cosingleton and a b.d.o. while a four-element algebra $(0, 1, A, A')$ is a b.f.o. Free objects are those which have a free system of generators and an algebra is a direct object if and only if it is the field of all subsets of a set. Injective objects are just the complete algebras (cf. [**21**; **8**]).

4. If $\mathcal{B}$ is the bicategory of left $\Lambda$-modules (cf. [**4**]), then $\Lambda$ is a

projective coseparator and $\mathrm{Hom}_Z(\Lambda, Q/Z)$ is an injective separator.

5. If $\mathcal{B}$ is the bicategory of compact Hausdorff spaces and continuous maps, then a one-point space is a singleton and a b.f.o. simultaneously and a closed interval is a b.d.o. Free joins are $\beta(\bigcup S_\alpha)$ (where $\bigcup S_\alpha$ is the disjoint union) and direct joins are Cartesian products, whence free objects are just $\beta(N_\alpha)$ with $N_\alpha$ discrete and direct objects are the Tichonov cubes. Projective objects are just the extremally disconnected ones (cf. [6; 16]) and injective objects are absolute retracts.

6. If $\mathcal{B}$ is the bicategory of normed linear spaces and linear operators with $\|T\| \leqq 1$ (injections being isometries into), then the real line is both b.f.o. and b.d.o. Free joins are $l_1$-direct sums i.e., spaces of functions $t \to x_t$ with $x_t \in X_t$ and $\|\{x_t\}\| = \sum \|x_t\| < \infty$, and direct joins are $m$-direct sums i.e., spaces of bounded functions $t \to x_t$ with $x_t \in X_t$ and $\|\{x_t\}\| = \sup \|x_t\|$. Thus, free objects are the spaces $l_1(N_\alpha)$ and direct objects are the spaces $m(N_\alpha)$. Projective objects are the same as free and injective ones are those with the binary intersection property i.e., those equivalent to spaces $C(S)$ with $S$ extremally disconnected.

The bicategory of normed linear spaces and all linear (continuous) operators (injections being bicontinuous) does not admit infinite free or direct joins. The spaces $l_1(N_\alpha)$ and $m(N_\alpha)$ are projective and injective, respectively. No characterization of projective and injective objects is known. For references see [2; 10; 14; 15; 19].

7. The bicategory of normal two-norm spaces and $\gamma$-$\gamma$-linear operators (cf. [1]) with $\|T\| = \sup\{\|Tx\| : \|x\| \leqq 1\} \leqq 1$ admits free and direct joins of countable sets of objects (defined as $l_1$-products and $m$-products). No nonzero object is injective (cf. [17]). The real line is injective in the category of $\gamma$-reflexive spaces and all $\gamma$-$\gamma$-linear maps (cf. [18]) but it is not injective for $\gamma$-$\gamma$-linear maps with $\|T\| \leqq 1$ because the number $\epsilon$ in Theorem 6 of [18] is indispensable.

The proofs and details will be published in [20].

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