ON VANISHING ALGEBRAS

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Let G be a locally compact group with left invariant Haar measure m. For any measurable subset S of G, define L_S to be that subset of $L^1(G)$ consisting of all functions which vanish (a.e.) on the complement of S. When L_S forms an algebra, we call it a *vanishing algebra.* It is known that when S is a semigroup l.a.e. (i.e., there exists a semigroup T in G such that $S=T$ locally almost everywhere), L_S is a vanishing algebra. The following theorem gives an answer to a problem formulated by A. Simon [2]:

THEOREM 1. *Suppose G is unimodular. If L_S is a vanishing algebra and S is contained in a σ-compact subset of G, then S is a semigroup a.e.*

COROLLARY 1. *Suppose G is compact. Then, if L_S is a vanishing algebra, S is a semigroup a.e.*

COROLLARY 2. *Suppose G is abelian and generated by some compact neighborhood of the identity element of G. Then, if L_S is a vanishing algebra, S is a semigroup a.e.*

The proof of Theorem 1 also gives the following more general and involved statement:

THEOREM 2. *Let L_S be a vanishing algebra. Suppose there exists a directed set $\{U_i, i\in I\}$ of symmetric neighborhoods of the identity element e with finite measures, having the property that for almost all the points x of S there exists a $j_x\in I$ such that $m(S\cap xU_i)$ and $m(x^{-1}U_i\cap S^{-1})$ are both $>m(U_i)/2$ as $i\geqq j_x$. Then S is a semigroup l.a.e. If, in addition, S is contained in a σ-compact subset of G, then S is a semigroup a.e.*

THEOREM 3. *If L_S is a self-adjoint vanishing algebra, then S is a group l.a.e. If, in addition, S is contained in a σ-compact subset of G, then S is a group a.e.*

THEOREM 4. *Let L_S be a vanishing algebra. If S is open, then S is a semigroup l.a.e. If, in addition, S is contained in a σ-compact subset of G, then S is a semigroup a.e.*

THEOREM 5. *If L_S is a maximal vanishing algebra, then S is a closed*

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semigroup l.a.e. If, in addition, S is contained in a σ-compact subset of G, then S is a closed semigroup a.e.

COROLLARY 3. *Let G be abelian and generated by some compact neighborhood of the identity element of G. If there exists a vanishing algebra L_S which is a maximal subalgebra in $L^1(G)$, then G is either the additive group of real numbers or the discrete integer group.*

REFERENCES

1. P. R. Halmos, *Measure theory*, New York, Van Nostrand, 1950.

2. A. B. Simon, *Vanishing algebras*, Trans. Amer. Math. Soc. 92 (1959), 154–167.

3. ———, *On the maximality of vanishing algebras*, Amer. J. Math. 81 (1959) 613–616.

4. A. Beck, H. H. Corson and A. B. Simon, *The interior points of the product of two subsets of a locally compact group*, Proc. Amer. Math. Soc. 9 (1958), 648–652.

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