ON THE EXISTENCE OF A SMALL CONNECTED OPEN SET WITH A CONNECTED BOUNDARY

BY F. BURTON JONES[1]

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In a connected, locally connected, locally compact metric space with no local separating point it is rather easy to construct an arbitrarily small connected open set whose boundary is a subset of an arbitrary small continuum lying in its complement. In fact such sets form a topological basis for the space. However, it seems to be much more difficult to construct small connected open sets whose boundaries are connected. The author constructed such open sets (substituting something weaker for local compactness) [1, Theorem 33] in certain special plane-like spaces but efforts at that time to generalize the theorem failed. Now with the help of the partitioning technique (brick partitioning, in particular) the construction may be carried out successfully.[2]

LEMMA. *Suppose that* (1) *U is a connected open proper subset of the connected, locally connected, compact metric space M such that $\overline{U}=M$,* (2) *p is a point of U such that $M-p$ is connected and* (3) *no point of $M-p$ is a local separating point of M. Then if ϵ is a positive number, there exists a connected open point set V such that* (1) *$p\in V\subset\overline{V}\subset U$,* (2) *$M-\overline{V}$ is connected and* (3) *if $x\in M-V$, $d(x, V)<\epsilon$.*

INDICATION OF PROOF. Let F denote the boundary of U. Without loss of generality we shall assume that 3ϵ is less than $d(p, F)$. Being compact and locally connected, M has property S. By Theorem 8 of [2] there exists a sequence $G_1, G_2, \cdots$ such that G_i is a brick $(1/i)$-partitioning of M and G_{i+1} is a refinement of G_i. Let B_i denote the subcollection of elements g of G_i such that $\bar{g}\cdot F\neq 0$ and let H_i denote the subcollection of G_i consisting of the elements of B_i together with all other elements of G_i which are separated from p by $\overline{B}_i^*$.

There exists a value of i such that each point of $\overline{H}_i^*$ is within $\epsilon/4$ of F. For suppose on the contrary that for each i, $\overline{H}_i^*$ contains a point q_i such that $d(q_i, F)\geqq\epsilon/4$. Let us suppose that $\{q_i\}$ converges to q (for certainly some subsequence converges). Since $d(q, F)\geqq\epsilon/4$, q belongs to U and there is an arc pq from p to q lying in U. Now let $i(q)$ be a value of i large enough so that if $g_1, g_2\in G_i$, $\bar{g}_1\cdot pq\neq 0$ and $\bar{g}_2\cdot F\neq 0$,

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then $\bar{g}_1 \cdot \bar{g}_2 = 0$. Obviously the interior of the sum of the closures of elements of $G_{i(q)}$ whose closures intersect pq does not intersect $\overline{H}_i^*$ for $i \geqq i(q)$. But since this interior is a connected open set containing $p+q$, it contains q_i for infinitely many values of i. This is a contradiction.

Furthermore, letting K_i denote $G_i - H_i$, there exists a value of i such that every point of F is within $\epsilon/4$ of $\text{int}(\overline{K}_i^*)$. Let i_1 be the larger of these two values of i and let V_1 denote $\text{int}(\overline{K}_{i_1}^*)$.

Clearly V_1 is a connected open subset of U containing p and $M_1 = \overline{V}_1$ is a subcontinuum of U having property S. Let $C_1, C_2, \cdots, C_{n_1}$ denote the components of $M - M_1$. Since the partitioning is a *brick* partitioning, these components cover F and their closures are non-intersecting. Since p does not separate M, there exists an arc ab in $M - p$ irreducible from $\overline{C}_1$ to $\overline{C}_2 + \overline{C}_3 + \cdots + \overline{C}_{n_1}$. Let Q denote the component of $V_1 - p$ containing $ab - (a+b)$; $\overline{Q}$ has property S. Since p does not separate $\overline{Q}$ and no point of $\overline{Q} - p$ is a local separating point of $\overline{Q}$, no pair of points of $\overline{Q}$ separates $\overline{Q}$. By Theorem 17 of [2] there exists an arc T' from a to b lying in $\overline{Q} - p$ which does not separate $\overline{Q}$. Let T_1 denote a subarc of T' irreducible from $\overline{C}_1$ to $\overline{C}_2 + \overline{C}_3 + \cdots + \overline{C}_{n_1}$.

The arc T_1 does not separate $\overline{Q}$ and lies (except for its endpoints) in Q. In fact $p + (Q - T_1)$ is connected. To see that this is true, let x denote a point of $Q - T'$. In $\overline{Q} - T'$ there exists an arc xp. There exists an integer $j > i_1$ such that (1) if $g_1, g_2 \in G_j$, $\bar{g}_1 \cdot xp \neq 0$ and $\bar{g}_2 \cdot T' \neq 0$, then $\bar{g}_1 \cdot \bar{g}_2 = 0$, and (2) if $g_1, g_2 \in G_j$, $\bar{g}_1 \cdot (x+p) \neq 0$ and $\bar{g}_2 \cdot (M - V_1) \neq 0$, then $\bar{g}_1 \cdot \bar{g}_2 = 0$. The interior I of the sum of the closures of the elements of G_j lying in V_1 whose closures intersect xp is a connected open subset of V_1 containing no point of T' and no point of the boundary of V_1. But I contains $p+x$. In I there is an arc px. But px is obviously a subset of $p + (Q - T')$. So $p + (Q - T')$ is connected. Since no point of $M_1 - p$ is a local separating point of M_1, every point of an arc in M_1 is a boundary point of the arc relative to M_1. Hence $p + (Q - T_1)$ is connected.

Clearly $U_1 = V_1 - T_1$ is connected and $M_1 = \overline{U}_1$. Let F_1 be the boundary of U_1.

Now the entire process may be repeated using U_1, M_1, and F_1 for U, M, and F respectively but being sure that i_2 is a value of i such that each point of $\overline{H}_i^*$ is within $\epsilon/4n_1$ of F_1 (n_1 was the number of the components C_i). The fact that p does not separate M need not be inherited by M_1. By taking T_2 to join the boundaries of different components of $M - M_2$, the number of these components is again reduced by at least one. Hence in n_1 (or less) steps V_{n_1} is the V called for in the lemma.

THEOREM. *Suppose that S is a connected, locally connected, locally compact metric space which contains no local separating point. Then if p is a point of an open subset R of S, there exists a connected open set D such that $p \in D \subset R$ and the boundary of D is connected.*

PROOF. By Theorem 2.4 of [3] there exists a connected, uniformly locally connected open proper subset U of R such that U contains p and $\overline{U}$ is compact. Let F denote the boundary of U and $M = U + F$. Hence by the lemma there exists a connected open set V_1 such that (1) $p \in V_1 \subset \overline{V}_1 \subset U$, (2) $M - \overline{V}_1$ is a connected open (rel M) subset of M containing F and (3) if x is a point of $M - V_1$ then $d(x, V_1) < 1$. Consider the decomposition space M_1 of M in which the only non-degenerate element P_1 is $\overline{V}_1$. Now P_1 does not separate M_1 and no other point of M_1 is a local separating point of M_1 relative to M_1. We can now reapply the lemma to get V_2 with the properties of V of the lemma so that if x is a point of $M_1 - V_2 (= M - V_2^*)$, then $d(x, V_2^*) < 1/2$ (here V_2^* is a subset of S and d is the distance function for S). This process may be continued so that $p \in V_1 \subset \overline{V}_1 \subset V_2^* \subset \overline{V}_2^* \subset \cdots$ where if x is a point of $M - V_n^*$, then $d(x, V_n^*) < 1/n$. Since $M - \overline{V}_n^*$ is connected, $\Pi(M - V_n^*)$ is a continuum containing F. Furthermore it is the boundary of ΣV_n^*. So ΣV_n^* is the required connected open subset D of R containing p.

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UNIVERSITY OF NORTH CAROLINA