A NEW PROOF AND AN EXTENSION OF HARTOG'S THEOREM[1]

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Let R denote n dimensional real euclidean space and let Ω_0 be a shell in R, by this we mean that there exist open sets Ω_1, Ω_2 where Ω_1 is relatively compact and has its closure contained in Ω_2, and $\Omega_0 = \Omega_2 -$closure Ω_1. Call Γ_j the boundary of Ω_j. Let $D=(D_1, \cdots, D_r)$ be a sequence of linear partial differential operators with constant coefficients on R with $r>1$. For a function f on R we write $Df=0$ if $D_j f=0$ for $j=1, 2, \cdots, r$. We want to determine the conditions on D in order that the following property should hold: If f is an indefinitely differentiable function on Ω_0 with $Df=0$ then there exists a unique indefinitely differentiable function h on Ω_2 with $Dh=0$ and $h=f$ on Ω_0. Hartog's theorem asserts that such an extension of f is possible if R is complex euclidean space of complex dimension $n/2=m>1$ and Ω_1 and Ω_2 are topological balls, and $D_j=\partial/\partial x_{2j-1}+i\,\partial/\partial x_j$ for $j=1, 2, \cdots, m$ where $x=(x_1, \cdots, x_n)$ are the coordinates on R. An extension of Hartog's theorem has been found by S. Bochner in [1] by a different method.

We can find a function g defined and C^∞ on Ω_2 such that $g=f$ on Ω_0 except on an arbitrarily small neighborhood $N(\Gamma_1)$ in Ω_0. (We choose $N(\Gamma_1)$ so small that its closure does not meet Γ_2.) Call $\Omega_3=\Omega_1\cup N(\Gamma_1)$. We have $Dg=0$ on $\Omega-\Omega_3$. We set $g_j=D_j g$, so g_j are C^∞ and have their supports in the closure of Ω_3; in particular the g_j are of compact support. For any j, k,

$$D_k g_j = D_j g_k \tag{1}$$

since both sides are equal to $D_k D_j g$ in Ω_3 and zero outside.

Next we take the Fourier transforms: Call P_k the Fourier transform of D_k and G_k that of g_k; P_k is a polynomial and G_k an entire function of exponential type on C (complex n-space); the exponential type of G_k is determined by the convex hull K of Ω_3. Moreover, G_k decreases on the real part of C faster than the reciprocal of any polynomial (see [5]). Relation (1) becomes

$$P_k(z)G_j(z) = P_j(z)G_k(z). \tag{2}$$

[1] Work supported by ONR 432 JLP.

We introduce now the assumption (α). *For any j, k, P_j and P_k are relatively prime.*

Under this it follows from (2) that G_k is divisible by P_k in the ring of entire functions. Thus there exists an entire function H such that

$$H = G_k/P_k \tag{3}$$

for all k. Using results of Malgrange and of the author (see [1]) it follows that H is again an entire function of exponential type which decreases on the real part of C faster than the reciprocal of any polynomial. Moreover the exponential type of H satisfies the conditions to make the Fourier inverse transform h of H have its support in K. In addition, $D_j h = g_j$.

We study now the support of h: $h=0$ outside K and $D_j h = g_j = 0$ outside Ω_3. To see what this means for the support of h, let us consider the example: $n=2m$, (Ω_1 and Ω_2 topological balls, $D_j = \partial/\partial x_j + i\partial/\partial x_{j+m}$ for $j=1, 2, \cdots, m$). Then h is holomorphic on $R-\Omega_3$ and $h=0$ outside Ω_3. If we choose Ω_3 (as we may) so that its boundary is a differentiable sphere, then it is easy to see by analytic continuation that $h=0$ outside Ω_3. Thus we are led to the assumptions.

(β) *We can choose $N(\Gamma_1)$ in such a way that the following unique continuation property holds: If a is C^∞ on R, and $a=0$ outside K, and $D_j a=0$ outside Ω_3 for $j=1, 2, \cdots, r$, then $a=0$ outside Ω_3.*

Using assumption (β) we see that we may assume that $h=0$ outside Ω_3.

Now, let us set $\tilde{f} = g-h$. Then clearly

(a) $\tilde{f}$ is C^∞ on Ω_2,

(b) $\tilde{f}=f$ on $\Omega_2 - \Omega_3$,

(c) $D_j\tilde{f} = D_j g - D_j h = g_j - g_j = 0$ on Ω_2.

Thus $\tilde{f}$ is an extension of f. It is clear that $\tilde{f}$ is unique, for if there were two extensions say $\tilde{f}$ and $\tilde{f}_1$ then $\tilde{f}-\tilde{f}_1$ would have its support in the closure of Ω_3 in particular $\tilde{f}-\tilde{f}_1$ would be of compact support which is incompatible with the fact that $D_j(\tilde{f}-\tilde{f}_1)=0$.

(γ) *Neighborhoods $N(\Gamma_1)$ satisfying assumption (β) can be chosen arbitrarily small.*

By the uniqueness property described above (or using the unique continuation property directly) we see that $\tilde{f}=f$ on Ω_0.

Thus we have the

THEOREM. *Under assumption (α), (β), (γ) any function f which is C^∞ on Ω_0 and satisfies $Df=0$ on Ω_0 possesses a C^∞ extension $\tilde{f}$ to Ω_2 with $Df=0$ on Ω_2.*

REMARKS. 1. It can be shown that the assumptions (α), (β), and (γ) are necessary for the theorem.

2. A similar result holds for distribution solutions of D.

3. For $r=1$ there are theorems of the above type for some D for C^∞ solutions but for no D for distribution solutions (because of the existence of a fundamental solution).

4. The above theorem and proof can be generalized to systems of convolution equations with invertible kernels of compact support (see [3]).

5. The above method applies only to extending f over compact subsets of Ω_2. In case $\Gamma_1 \cap \Gamma_2$ is not empty the problem of extending f requires new considerations. We hope to return to this question at a future date.

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