ON A NEW FUNCTIONAL TRANSFORM IN ANALYSIS: THE MAXIMUM TRANSFORM

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1. Introduction. In the study of mathematical economics and operations research, we encounter the problem of determining the maximum of the function

$$F(x_1, x_2, \cdots, x_N) = f_1(x_1) + f_2(x_2) + \cdots + f_N(x_N) \tag{1}$$

over the region R defined by $x_1+x_2+ \cdots +x_N=x$, $x_i \geqq 0$. Under various assumptions concerning the f_i, this problem can be studied analytically; cf. Karush [1; 2], and it can also be treated analytically by means of the theory of dynamic programming [3].

It is natural in this connection to introduce a "convolution" of two functions f and g, $h=f * g$, defined by

$$h(x) = \max_{0 \leqq y \leqq x} [f(y) + g(x - y)]. \tag{2}$$

For purposes of general study, it is more convenient to introduce instead the convolution $h=f \otimes g$ defined by

$$h(x) = \max_{0 \leqq y \leqq x} [f(y)g(x - y)]. \tag{3}$$

It is easy to see that the operation $\otimes$ is commutative and associative provided that all functions involved are nonnegative. By analogy with the relation between the Laplace transform and the usual convolution, $\int_0^x f(y)g(x-y)dy$, it is natural to seek a functional transform

$$M(f) = F \tag{4}$$

with the property that

$$M(f \otimes g) = M(f)M(g), \tag{5}$$

that is,

$$H(z) = F(z)G(z) \tag{6}$$

where H, F, G are the transforms of h, f, g respectively.

We shall show that M exists and has a very simple form. In addition, M^{-1} has a very simple and elegant representation in a number of cases. More detailed discussions and extensions will be presented subsequently.

2. The maximum transform. Let a transform (1.4) be defined by the equation

$$F(z) = \max_{x\geqq 0} \left[e^{-xz}f(x)\right], \qquad z \geqq 0. \tag{1}$$

It will be assumed that $f(x)$ is continuous and nonnegative for $x\geqq 0$. Furthermore, since $F(z)$ is unchanged when f is replaced by its monotone envelope, we shall consider (1) only for monotone nondecreasing f.

It is now a straightforward matter to prove (1.5) by the method used in the usual convolution. We have

$$\begin{aligned} H(z) &= \max_{x\geqq 0}\left[e^{-xz}\max_{0\leqq y\leqq x}\left[f(y)g(x-y)\right]\right] \\ &= \max_{x\geqq 0}\ \max_{0\leqq y\leqq x}\left[e^{-xz}f(y)g(x-y)\right] = \max_{y\geqq 0}\ \max_{x\geqq y}\left[\ \right] \\ &= \max_{y\geqq 0}\left[f(y)\max_{x\geqq y}\left[e^{-xz}g(x-y)\right]\right] \\ &= \max_{y\geqq 0}\left[e^{-yz}f(y)\max_{w\geqq 0}\left[e^{-wz}g(w)\right]\right] \\ &= \max_{y\geqq 0}\left[e^{-yz}f(y)\right]\cdot\max_{w\geqq 0}\left[e^{-wz}g(w)\right] = F(z)G(z) \end{aligned} \tag{2}$$

as desired.

To ensure the existence of $F=M(f)$ for $z>0$, it is sufficient to assume that f satisfies a relation of the form $f(x)=O[x^c]$ for $x\geqq 0$ where $c\geqq 0$. The transform f is decreasing and continuous for $z>0$; if $c=0$, this holds for $z\geqq 0$.

3. Inverse operator. The determination of the existence and uniqueness of M^{-1} is of some complexity, and at this time we shall consider only special cases. If for $z>0$, the maximum of $f(x)e^{-xz}$ can be found by differentiation, we have the maximizing value the equation $f'(x)-zf(x)=0$. Suppose that this equation possesses a unique solution $x=x(z)$ with $dx/dz\neq 0$ (and hence <0). For this value of x, we have $F(z)=e^{-xz}f(x)$. Differentiating this relation with respect to x, we have

$$F'(z)\frac{dz}{dx} = (f'(x) - zf(x))e^{-xz} - xf(x)e^{-xz}\frac{dz}{dx} = -\,xf(x)e^{-xz}\frac{dz}{dx}\,. \tag{1}$$

Hence,

$$x = -\,F'(z)/F(z), \quad \text{or} \quad F'(z) + xF(z) = 0. \tag{2}$$

But this is precisely the relation which gives the z minimizing $F(z)e^{xz}$, for fixed x. Hence, we have

$$f(x) = \min_{z \geqq 0} e^{xz}F(z), \tag{3}$$

the required inversion relation.

A simpler way to obtain this relation is the following. By (2.1), we have, for $x \geqq 0$,

$$F(z) \geqq e^{-xz}f(x), \tag{4}$$

whence $F(z)e^{xz} \geqq f(x)$. If there is a one-to-one correspondence between x and z values, we have $\min_{z \geqq 0} F(z)e^{zx} \geqq f(x)$, with equality for one value, whence (3).

4. Application. Let

$$f(x) = \max_R [f_1(x_1)f_2(x_2) \cdots f_N(x_N)], \tag{1}$$

where R is as in (1.1). Then, inductively,

$$M(f) = \prod_{i=1}^{N} M(f_i), \quad \text{or} \quad F(z) = \prod_{i=1}^{N} F_i(z), \tag{2}$$

whence formally

$$f(x) = \min_{z \geqq 0} \left[e^{xz} \prod_{i=1}^{N} F_i(z) \right]. \tag{3}$$

Similarly, if we have a "renewal" equation

$$f(x) = a(x) + \max_{0 \leqq y \leqq x} [f(y)g(x - y)], \tag{4}$$

we have a formal solution

$$f(x) = \min_{z \geqq 0} \left[\frac{e^{xz}A(z)}{1 - G(z)} \right], \tag{5}$$

where $A = M(a)$, $G = M(g)$.

References

1. W. Karush, *A queuing model for an inventory problem*, Operations Res. vol. 5 (1957) pp. 693–703.

2. ———, *A general algorithm for the optimal distribution of effort*, to appear.

3. R. Bellman, *Dynamic programming*, Princeton, New Jersey, Princeton University Press, 1957.

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