CONICAL SINGULAR POINTS OF DIFFEOMORPHISMS

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1. Introduction. The Schoenflies extension Λ_ϕ of a differentiable mapping ϕ, constructed in the proof of Theorem 2.1 of [1], has at most a differential singularity of *conical* type (to be defined). This fact has far-reaching consequences which are reflected in the theorems of [2]. Theorem 1.1 below is one of these consequences. No proof of Theorem 1.1 is given here.

Let S be an $(n-1)$-sphere in a euclidean n-space E and let JS be the closed n-ball in E bounded by S.

THEOREM 1.1. *Let z be an arbitrary point of S. A real analytic diffeomorphism f of S into E admits a homeomorphic extension, F, defined over a set $Z \cup z$, where Z is some open neighborhood of $JS-z$, and $F|Z$ is a real analytic diffeomorphism of Z into E.*

This extension F of f defines an analytic diffeomorphism of its domain of definition with z deleted, and a homeomorphism with z included. F has no singularity on the interior of S, or on S, except at most at z.

We continue with a detailed exposition leading to a proof of Theorem 2.1.

NOTATION. Let E be the euclidean n-space of points (or vectors) x with rectangular coordinates $(x_1, \cdots, x_n)$. Let $\|x\|$ be the distance of x from the origin O. Set

$$S = \{x \mid \|x\| = 1\}. \tag{1.1}$$

If M is a topological $(n-1)$-sphere in E, $\mathring{J}M$ shall denote the open interior of M. The complement of a subset Y of E will be denoted by CY. We use *diff* as an abbreviation of diffeomorphism.

A C_z^m-diff, $m>0$. Let $x \to G(x)$ be a homeomorphism into E of an open neighborhood X of a point $z \in E$; if $G|(X-z)$ is a C^m-diff into E, G will be called a C_z^m-diff of X into E.

An admissible cone K_z. Let K_z be a closed n-cone in E with vertex z, and with sections orthogonal to A which are closed $(n-1)$-balls whose centers are on A. The cone K_z is determined by z, A and any one of its orthogonal sections meeting $A-z$.

A conical point z of G. Let G be a C_z^m-diff into E of an open neighborhood X of z. The point z will be said to be a conical point of G and

K_z *a cone of singular approach* to z if there exists a C^m-diff ζ into E of some open neighborhood $U \subset X$ of z such that

$$G(x) = \zeta(x) \qquad (x \in U - K_z). \tag{1.2}$$

On the supposition that z is a conical point of G we prove the following lemma.

LEMMA 1.1. (i) *If μ is a C^m-diff into X of an open neighborhood Y of a point y such that $\mu(y) = z$, then $G\mu$ is a C_y^m-diff of Y into E with conical point y.*

(ii) *If θ is a C^m-diff of $G(X)$ into E, then θG is a C_z^m-diff of X into E with conical point z.*

PROOF OF (i). Suppose that G is represented on $U - K_z$ as in (1.2). Let $W \subset Y$ be an open neighborhood of y so small that $\mu(W) \subset U$, and for some admissible cone K_y

$$\mu(W - K_y) \subset U - K_z. \tag{1.3}$$

Put $\mu | W = \mu_1$. Then $\zeta\mu_1$ is a C^m-diff of W into E, and it follows from (1.2) and (1.3) that

$$(G\mu)(x) = (\zeta\mu_1)(x) \qquad (x \in W - K_y). \tag{1.4}$$

This partial representation (1.4) of $G\mu$ shows that y is a conical point of the C_y^m-diff $G\mu$. This establishes (i).

The proof of (ii) is immediate.

2. **The principal theorem.** In [1] there is given a C^m-diff ϕ into E of a "shell" neighborhood δ_a of S such that ϕ carries points of δ_a interior to S into points of E interior to the manifold $\phi(S)$, and it is shown (see [1, Theorem 2.1]) that there exists an open neighborhood $U \subset \delta_a$ of S, a point $z \in \mathring{J}S$ and a C_z^m-diff Λ_ϕ of $U \cup \mathring{J}S$ into E which extends $\phi | U$. The construction of Λ_ϕ is carried through in [1] for the case in which ϕ is *special*, in the sense that ϕ reduces to the identity in the neighborhood of a point Q of S. In this paper we supplement Theorem 2.1 of [1] by proving the following.

THEOREM 2.1. *The C_z^m-diff Λ_ϕ, as constructed in [1] for a "special" C^m-diff ϕ, has z as conical point.*

To prove this theorem we review the necessary parts of the construction of Λ_ϕ in [1].

The relevant subsets of E. Let K be the open n-cube [1, p. 273]

$$K = \{x \mid -1 < x_i < 1;\ i = 1, \cdots, n\}. \tag{2.1}$$

Let K' be the subrectangle of K on which $x_n < 0$. Subrectangles $H' \supset L' \supset G'$ of K' are introduced with faces parallel to those of K', of which H' and L' are open and G' closed, while

$$\text{(2.2)} \qquad \mathrm{Cl}\, H' \subset K', \qquad \mathrm{Cl}\, L' \subset H'.$$

Let $D = \{x \mid -1 < x_i < 9;\ i = 1, \cdots, n\}$ and set $P = (8, 0, \cdots, 0)$.

A radial mapping R of E onto E. R is defined by the equations

$$\text{(2.3)} \qquad y_1 - 8 = \frac{x_1 - 8}{2}; \qquad y_j = \frac{x_j}{2} \qquad (j = 2, \cdots, n)$$

and leaves P fixed. If R^r is the r-fold iterate of R and R^0 the identity, the space E admits a trivial partition

$$\text{(2.4)} \qquad E = \left[\bigcup_{r=0}^{\infty} R^r(K) \right] \cup P \cup A \qquad \text{(Cf. (5.1) of [1])}$$

provided A is suitably chosen.

The mapping T. If B is a bounded subset of E, Int B shall denote the smallest n-rectangle Π in E with faces parallel to the coordinate $(n-1)$-planes and such that $\Pi \supset B$. In §6 of [1], a C^∞-diff T of E onto E is defined. For us the essential properties of T are that

$$\text{(2.5)} \qquad RT(\overline{K}) \cap T(\overline{K}) = \emptyset, \qquad T(\overline{K}) \subset \mathrm{Int}(\overline{K} \cup R\overline{K}).$$

One sets $T_{r+1} = R^r T$, $r = 0, 1, \cdots$.

The cone K_P. Let K_P be the smallest admissible cone with vertex P, with axis the segment of the x_1-axis on which $x_1 \leq 8$, and with $K_P \supset \mathrm{Int}(\overline{K} \cup R\overline{K})$. One sees that

$$\text{(2.6)} \qquad K_P \supset T_r(K) \cup G' \qquad (r = 1, 2, \cdots).$$

The contraction **a**. This is a C^∞-diff of D onto H' which leaves L' pointwise invariant [1, p. 274]. We infer that $\mathbf{a}(P) \in H' - \mathrm{Cl}\, L'$.

The reflection t. The point Q is the intersection of the positive x_n-axis with S. Let S_Q be an $(n-1)$-sphere with center Q and diameter $\rho < 1$, so small that $\phi \mid JS_Q$ reduces to the identity. Let t be the reflection of $E - Q$ in S_Q [1, p. 272].

The C^m-diff ω. The domain of definition of ω includes $H' - G'$, and so is an open neighborhood of $\mathbf{a}(P)$. Cf. p. 273 of [1].

The mapping ω_e. By Lemma 5.1 of [1], the domain of ω_e includes A, and $\omega_e(x) = x$ on A.

The mapping σ. By Lemma 7.2 of [1] σ is a C_P^m-diff of CG' into E. By this lemma and (2.6), $\sigma(x) = \omega_e(x)$ for $x \in CK_P$. Since $CK_P \subset A$ by (2.4), $\sigma(x) = x$ for $x \in CK_P$. Hence P is a conical point of σ.

The mapping $\lambda_\omega \mathbf{a}$. For present purposes $\lambda_\omega \mathbf{a}$ is a mapping for which (cf. (3.6) of [1])

$$(2.7) \qquad (\lambda_\omega \mathbf{a})(x) = \omega(\mathbf{a}(\sigma^{-1}(x))) \qquad (x \in \sigma(D - G')).$$

We verify that P is a conical point of $\lambda_\omega \mathbf{a}$. The domain of x in (2.7) is open. It contains P since $P \in D - G'$ and $\sigma(P) = P$. Now σ^{-1} maps $\sigma(D - G')$, as a C_P^m-diff, onto $D - G'$, with P a conical point of σ^{-1}. Moreover $\mathbf{a}(D - G') = H' - G'$, while ω operates as a C^m-diff on $H' - G'$. Returning to (2.7) observe that $\lambda_\omega \mathbf{a}$ defines a C_P^m-diff of $\sigma(D - G')$ into E. It follows from Lemma 1.1 (ii) and (2.7) that P is a conical point of $\lambda_\omega \mathbf{a}$.

COMPLETION OF PROOF OF THEOREM 2.1. In accord with the line following (7.19) of [1] and line -13 on p. 275 in [1], $z = t(\mathbf{a}(P))$. By (7.22)'' of [1]

$$(2.8) \qquad \Lambda_\phi(x) = t(\lambda_\omega(t(x))) \qquad (x \in t(H'))$$

(cf. [1, lines 2–3, p. 287]) so that if one sets $\mu(x) = \mathbf{a}^{-1}(t(x))$ for $x \in t(H')$

$$(2.9) \qquad \Lambda_\phi(x) = [t(\lambda_\omega \mathbf{a})\mu](x) \qquad (x \in t(H')).$$

We now apply Lemma 1.1. The C^∞-diff $x \to \mu(x)$ of $t(H')$ into E maps z into P, since $z = t(\mathbf{a}(P))$. From Lemma 1.1 and (2.9) we can then infer that z is a conical point of Λ_ϕ, since P is a conical point of $\lambda_\omega \mathbf{a}$.

This establishes Theorem 2.1.

The generality of singularities of conical type is evidenced by the following theorem. Cf. [2].

THEOREM 2.2. *Let F be an arbitrary C_z^m-diffeomorphism into E of an open subset $X \subset E$. There exists a C_z^m-diffeomorphism F^* of X into E for which z is a conical point and which is such that $F^*(x) = F(x)$ except at most in an arbitrarily small prescribed neighborhood of z.*

REFERENCES

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2. ———, *Schoenflies extensions without interior differential singularity*, Ann. of Math., to appear.

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