ON A PROBLEM OF P. A. SMITH[1]

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Communicated by Deane Montgomery, March 31, 1961

1. **Introduction.** Throughout this note, Z_2 denotes the group of integers mod 2 and cohomology means the Alexander-Wallace-Spanier cohomology with coefficients in Z_2. By a *cohomology projective n-space* we mean a compact Hausdorff space Y whose cohomology ring $H^*(Y)$ is isomorphic to that of the real projective n-space. In [2], Smith proved that if Z_2 acts effectively on the real projective n-space such that the fixed point set $F(Z_2)$ is nonempty, then $F(Z_2)$ has exactly two components A_1 and A_2, where A_i is a cohomology projective n_i-space ($i=1, 2$) and $n_1+n_2=n-1$. Smith then asked whether the result is true if the real projective n-space is replaced by a cohomology projective n-space. The purpose of this note is to give a positive answer to the question.

We wish to point out that the inclusion of ring structure in the definition of a cohomology projective n-space is indispensable as we may see from the following example. Let Y be the one-point union of a 1-sphere S^1 and a 2-sphere S^2. Clearly $H^*(Y)$ as a group is the same as the cohomology group of a projective plane. Let T be a generator of Z_2 and define the action of T on Y such that on S^i it is the reflexion with respect to the diameter passing through the point of contact. Then the fixed point set consists of three isolated points.

2. **A construction.** The proof of Smith's theorem in [2] has used the fact that a projective n-space admits an n-sphere as its two-folded covering space. It is therefore quite natural to expect that a cohomology projective n-space Y admits a cohomology n-sphere as its two-folded covering space. In the following we give a construction of such a cohomology n-sphere which is very similar to the construction of a covering space of a pathwise connected, locally pathwise connected, and locally pathwise simply connected space, with the dual of $H^1(Y)$ playing the role of fundamental group.

Let Y be a connected compact Hausdorff space and let $\alpha \in H^1(Y)$ be a nonzero element. Let f: $Y^2 \to Z_2$ be a 1-cocycle representing α; then there exists an open covering $\mathcal{V}$ of Y such that

$$f(y_0, y_2) = f(y_0, y_1) + f(y_1, y_2) \quad \text{whenever} \quad y_0, y_1, y_2 \in V \in \mathcal{V}.$$

[1] The research of this paper is supported by U. S. Army Research Office under Contract No. DA 36-034-ORD-2970. The author also wishes to express his debt to Professor C. T. Yang for his helpful suggestions.

Fix a point $b\in Y$. By a *$\mathcal{V}$-chain with base point b* we mean a finite sequence $(y_i)_{i=0}^n$ of points of Y such that $y_0=b$ and $\{y_{i-1}, y_i\}$ is contained in some $V\in\mathcal{V}$ for all $i=1, 2, \cdots, n$, the set of all $\mathcal{V}$-chains with base point b is denoted by $\mathfrak{X}$. Two $\mathcal{V}$-chains $(y_i)_{i=0}^n$ and $(y_j')_{j=0}^m$ are said to be equivalent if

$$\text{(i)} \qquad y_n = y_m',$$

$$\text{(ii)} \quad \sum_{i=1}^n f(y_{i-1}, y_i) = \sum_{j=1}^m f(y_{j-1}', y_j').$$

The quotient set of $\mathfrak{X}$ under this equivalence relation is denoted by X and the equivalence class of $(y_i)_{i=0}^n$ is denoted by $[y_i]_{i=0}^n$.

Now we topologize X as follows. Let $x=[y_i]_{i=0}^n\in X$ and $\mathfrak{B}(y_n)$ be a base of neighborhood of y_n such that every $B(y_n)\in\mathfrak{B}(y_n)$ is contained in some $V\in\mathcal{V}$. To each $B(y_n)\in\mathfrak{B}(y_n)$, we define

$$B^*(x) = \left\{[y_j']_{j=0}^m \,\middle|\, y_m' \in B(y_n),\ \sum_{i=1}^n f(y_{i-1}, y_i) + f(y_n, y_m') + \sum_{j=1}^m f(y_{j-1}', y_j') = 0\right\}.$$

It is easily verified that X is made a Hausdorff space with

$$\mathfrak{B}(x) = \{B^*(x) \mid B(y_n) \in \mathfrak{B}(y_n)\}$$

as a base of neighborhoods of x.

Define a map $\pi: X\to Y$ by $\pi([y_i]_{i=0}^n)=y_n$, it is straightforward to verify that π is well-defined and is a local homeomorphism of X onto Y.

Obviously, to each $y\in Y$, $\pi^{-1}(y)$ has at most two points. We now claim that it has exactly two points. To see this, it suffices to consider the case when $y=b$. Since $[b]$ is one point of $\pi^{-1}(b)$, all we have to do is to exhibit a $\mathcal{V}$-chain $(y_i)_{i=0}^n$ with $y_0=y_n=b$ and $\sum_{i=1}^n f(y_{i-1}, y_i)=1$. Suppose such a chain does not exist, then we can define a 0-cochain g: $Y\to Z_2$ by $g(y)=\sum_{i=1}^n f(y_{i-1}, y_i)$, where $(y_i)_{i=0}^n$ is any $\mathcal{V}$-chain with base point b with $y_n=y$. Such a chain exists in view of the connectedness of Y and g is clearly well-defined. Now if $\{y, y'\}\in V\in\mathcal{V}$, we have

$$g(y') - g(y) = \sum_{i=1}^n f(y_{i-1}, y_i) + f(y, y') - \sum_{i=1}^n f(y_{i-1}, y_i) = f(y, y').$$

But this means $f-\delta g$ has empty support, contradicting the assumption that $\alpha\neq 0$.

Now let T be the generator of Z_2 and define the action of T by exchanging the two points in $\pi^{-1}(y)$ for each $y\in Y$. We clearly obtain a free action of Z_2 on the compact Hausdorff space X with $Y=X/Z_2$.

Define a 0-cochain $h: X\to Z_2$ by

$$h([y_i]_{i=0}^n) = \sum_{i=1}^n f(y_{i-1}, y_i).$$

A similar argument as above shows that $\pi^*(\alpha)$ is the cohomology class of δh.

Suppose that now Y is a cohomology projective n-space and that α is the generator of the cohomology ring $H^*(Y)$. We claim that X is a cohomology n-sphere. As seen in [1], we have the exact Smith-Gysin sequence

$$\cdots \to H^k(Y) \xrightarrow{\pi^*} H^k(X) \xrightarrow{\tau_*} H^k(Y) \xrightarrow{\delta^*} H^{k+1}(Y) \to \cdots .$$

Since $\pi^*(\alpha)=0$ and π^* is a ring homomorphism, it follows that $\pi^*: H^k(Y)\to H^k(X)$ is trivial for all $k>0$. This is enough to conclude that

$$H^k(X) = \begin{cases} Z_2, & k = 0, n, \\ 0, & \text{otherwise.} \end{cases}$$

3. Main theorem.

THEOREM. *If Z_2 acts effectively on a cohomology projective n-space Y such that the fixed point set $F(Z_2)$ is nonempty, then $F(Z_2)$ has exactly two components A_1 and A_2 where each A_i is a cohomology projective n_i-space ($i=1$, 2) and $n_1+n_2=n-1$.*

PROOF. Let S be the generator of Z_2. In the construction of X given in the last section, we may choose the base point b in $F(Z_2)$ and we may assume that $\mathcal{V}$ is S-invariant (i.e. $S(V)\in\mathcal{V}$ for all $V\in\mathcal{V}$). It follows that S maps $\mathcal{V}$-chains with base point b into themselves or S induces a transformation on $\mathfrak{X}$. Observe that S also induces an automorphism S^* on $H^1(Y)$; hence we must have $S^*(\alpha)=\alpha$. It is easily seen that this fact implies that S maps equivalent $\mathcal{V}$-chains into themselves, in other words S induces a transformation $\hat{S}$ on the space X which is clearly compatible with π (i.e. $\pi\circ\hat{S}=S\circ\pi$). This means we have an action of the group $Z_2\times Z_2$ on a cohomology n-sphere X. The rest of the proof is word by word the same as given in [2].

REFERENCES

1. P. E. Conner and E. E. Floyd, *Fixed point free involutions and equivariant maps*, Bull. Amer. Math. Soc. vol. 66 (1960) pp. 416–441.

2. P. A. Smith, *New results and old problems in finite transformation groups*, Bull. Amer. Math. Soc. vol. 66 (1960) pp. 401–415.

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