LOCAL CONNECTIVITY IN HOMEOMORPHISM GROUPS

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Recently there has been increasing interest in the local connectivity of the group of all homeomorphisms of a manifold with boundary. Available tools of proof, however, seem to favor the case of a compact manifold [**1**; **2**] or else the use of the topology of uniform convergence of the group [**3**]. The present note extends such results to the groups of homeomorphisms of certain noncompact manifolds, furnished with the compact-open topology (also see [**4**]).

If X is a manifold with boundary, let $G(X)$ be the group of homeomorphisms of X, with compact-open topology. $G(X)$ is then a topological transformation group on X. Let the statement that a space X is (respectively) locally connected, locally contractible or locally n-connected be abbreviated by the phase "X is P_h", $h=1$, 2 or 3 respectively.

THEOREM. *Let X be a compact, connected, Hausdorff manifold with boundary,* dim $(X)>1$, *and let F be a finite set of nonboundary points of X. If $G(X)$ is P_h, $h=1$, 2 or 3, then $G(X-F)$ is P_h.*

In particular, this combines with the results of Hamstrom and Dyer [**1**] to show that if dim $(X)=2$ then $G(X-F)$ is locally contractible; and with the results of Hamstrom [**2**] to show that if dim $(X)=3$ then $G(X-F)$ is locally n-connected, for all n.

The following lemma will be used in the proof:

LEMMA. *Let X be a compact, connected, Hausdorff manifold with boundary,* dim $(X)>1$; *and let F be a finite set of nonboundary points of X. Then $G(X-F)$ is topologically isomorphic to $G(X, F)$ $=\{g\in G(X): g|F\in G(F)\}$.*

The proof of the lemma is an exercise in the compact-open topology; the hypothesis that X is a manifold is used in an application of the Jordan-Brouwer theorem. This proof is too long to be given here; it will appear elsewhere in another connection.

PROOF OF THE THEOREM. Induction will be used on the number of points of F. Let Y be the set of nonboundary points of X, and let $\{x_j\}$ be a sequence of distinct points of Y. Define $F_i=\bigcup_{j=1}^{i}\{x_j\}$ and $G_i=\{g\in G(X): g(x_j)=x_j \text{ if } x_j\in F_i\}$, with the relative topology.

(i) G_i is a principal fiber bundle over $Y-F_i$ with projection $p: G_i\to Y-F_i: g\to g(x_{i+1})$ and fiber G_{i+1}, for $i=0, 1, \cdots$. The proof of this fact uses the bundle structure theorem: G_{i+1} is a closed subgroup of G_i, and G_i will be a bundle over G_i/G_{i+1} if G_{i+1} has a local cross-section in G_i. Furthermore, if the map p is open, then G_i/G_{i+1} is homeomorphic to the domain of transitivity of G_i, namely $Y-F_i$. Choose a neighborhood N of x_{i+1} and a homeomorphism of N with a Euclidean space E^k. The translations of E^k provide a set of homeomorphisms of N, exactly one of which takes x_{i+1} to each $y\in N$. These maps are all extendible by the identity map outside N to homeomorphisms of $X-F_i$; the extensions provide a cross-section in G_i above N, and p is open.

(ii) If G_i is P_h, $h=1$, 2 or 3, then G_{i+1} is P_h; this is an instance of the general remark that, if F is the fiber of an arbitrary bundle E, F is P_h if E is P_h. The following argument, however, uses the instant notation. Choose a neighborhood N of x_{i+1} such that $p^{-1}(N)$ is homeomorphic to the product $N\times G_{i+1}$: rename the points of $p^{-1}(N)$ using this homeomorphism, and let $q: N\times G_{i+1}\to G_{i+1}$ be the coordinate projection. Let U be a neighborhood of 1 in G_{i+1}; $N\times U$ is a neighborhood of 1 in G_i.

If G_i is locally connected, choose a connected neighborhood V of 1, $V\subset N\times U$; then $q(V)\subset U$ is a connected neighborhood of 1 in G_{i+1}. If G_i is contractible, choose a neighborhood $V\subset N\times U$ of 1 and a contraction $H: V\times I\to N\times U$; then $\overline{H}: (V\cap G_{i+1})\times I\to U: (v, t)\to(q\circ H)(v, t)$ is a contraction of $V\cap G_{i+1}$. A similar construction works in case G_i is locally n-connected.

(iii) Define $G(X, F_i)=\{g\in G(X): g| F_i\in G(F_i)\}$ with the relative topology. Choose open neighborhoods N_j for each $x_j\in F_i$ such that $N_j\cap N_k=\emptyset$ if $j\neq k$; then $G_i=\{g\in G(X, F_i): g(x_j)\in N_j$ if $x_j\in F_i\}$ which is an open set in $G(X, F_i)$. Hence G_i is P_h, $h=1$, 2 or 3, iff $G(X,F_i)$ is also P_h.

(iv) By the lemma, $G(X, F_i)$ is topologically isomorphic to $G(X-F_i)$. Thus if $G(X)=G_0$ is P_h, $h=1$, 2 or 3, then G_i, $G(X, F_i)$ and finally $G(X-F_i)$ have been shown to be P_h.

REMARK. It may be possible to prove by similar methods a like theorem when boundary points are deleted or when X is not connected. The methods are worthless, however, if F_i is not discrete.

REFERENCES

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