A MINIMAL DEGREE LESS THAN $0'$

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Clifford Spector in [4] proved that there exists a minimal degree less than $0''$. J. R. Shoenfield in [3] asked: "Does there exist a minimal degree a such that $a \leqq 0'$?" We show that the answer to his question is yes! Our notation is that of [4].

We say that b strictly extends a if b and a are distinct sequence numbers, and if the sequence represented by b extends the one represented by a; we express this symbolically as SExt (b, a). If $\{a_0, a_1, a_2, \cdots\}$ is a sequence of sequence numbers such that for each i, a_{i+1} strictly extends a_i, then there is a unique function $f(n)$ such that for each i there is an m with the property that $\bar{f}(m) = a_i$; if $\{a_0, a_1, a_2, \cdots\} \subseteq S$, then we say $f(n)$ is a function associated with S. Spector in [4] obtained a function of minimal degree as the unique function associated with every member of a contracting sequence of sets of sequence numbers. Our construction is inspired by his, but it differs markedly from his in one respect: each one of our sets of sequence numbers will be recursively enumerable, whereas each one of his was recursive.

For each natural number c, let c^* be the unique, recursively enumerable set which has c as a Gödel number. There exists a recursive function $g(n)$ such that for each c, $g(c)$ is the Gödel number of the representing function of a recursive predicate $R_c(m, x)$ with the property that $x \in c^*$ if and only if $(Em)R_c(m, x)$. We define a recursive predicate $H(c, t, e, x, m, b, d)$ which is basic to our construction:

$$H(c, t, e, x, m, b, d) \equiv (i)_{i<2}(\mathrm{SExt}((x)_i, t) \;\&\; R_c((m)_i, (x)_i) \;\&\; T_1^1((x)_i, e, b, (d)_i)) \;\&\; U((d)_0) \neq U((d)_1).$$

We define a partial recursive function $Y(c, t, e)$:

$$Y(c, t, e) = \begin{cases} \mu x H(c, t, e, (x)_0, (x)_1, (x)_2, (x)_3) & \text{if } (Ex)H(c, t, e, (x)_0, (x)_1, (x)_2, (x)_3) \\ \text{undefined otherwise.} \end{cases}$$

We define a recursively enumerable set of sequence numbers denoted by $W(c, t, e)$: (a) $t \in W(c, t, e)$ if t is a sequence number; (b) if $u \in W(c, t, e)$ and if $Y(c, u, e)$ is defined, then $(Y(c, u, e))_{0,0} \in W(c, t, e)$ and $(Y(c, u, e))_{0,1} \in W(c, t, e)$; and (c) every member of $W(c, t, e)$ is

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obtained by an application of (a) followed by finitely many applications of (b). It is clear that there exists an effective procedure for computing a Gödel number of $W(c, t, e)$ from the triple (c, t, e). We define the recursive function $V(c, t, e)$ to be that function whose value for the triple (c, t, e) is equal to the result of applying this effective procedure to the triple (c, t, e).

We are ready to define four functions simultaneously by induction, $Q(i, j)$, $v(i)$, $u(i)$ and $t(i)$, where i and j are natural numbers. $Q(i, j)$ will take only 0 and 1 as values. The sequence $\{u(0), u(1), u(2), \cdots\}$ will consist of sequence numbers such that for each n, $u(n+1)$ will strictly extend $u(n)$; the unique function $h(n)$ associated with this sequence will have minimal degree.

Let q be a Gödel number of the set of all sequence numbers. Let $0(n)$ be the function which is everywhere 0. If t is a sequence number, let $w(t)$ be the least x such that $\mathrm{SExt}((x)_0, t)$, $\mathrm{SExt}((x)_1, t)$, $(x)_0$ does not strictly extend $(x)_1$, $(x)_1$ does not strictly extend $(x)_0$ and $(x)_0 \neq (x)_1$. We set $t(0)=0$ and $Q(i, 0)=1$ for all $i>0$. We set $u(0) = 2^{1+\overline{sg}(\{0\}^0(0))}$ if the latter expression is defined; otherwise we set $u(0)=2$. If $Y(q, u(0), 0)$ is defined, then we set $Q(0, 0)=1$ and $v(0) = (Y(q, u(0), 0))_0$; otherwise we set $Q(0, 0)=0$ and $v(0)=w(u(0))$.

Now suppose that $Q(i, s-1)$ has been defined for all i, and that $v(s-1)$ and $u(s-1)$ have also been defined, where $s>0$. Suppose further that $(v(s-1))_0$ and $(v(s-1))_1$ are distinct sequence numbers such that neither one strictly extends the other. Let $u(s)$ be the least one of $(v(s-1))_0$ and $(v(s-1))_1$ which is not strictly extended by $\prod_{i<v(s-1)} p_i^{1+\{s\}^0(i)}$, if the latter expression is defined; otherwise, let $u(s) = (v(s-1))_0$. Let $\{i \mid i < s, Q(i, s-1) = 1\} \cup \{s, s+1\} = \{i_1, i_2, \cdots, i_{r_s+1}\}$, where $i_1 < i_2 < \cdots < i_{r_s+1}$. Let $v_0^s = q$; and for each $k < r_s$, let $v_{k+1}^s = V(v_k^s, u(s), i_{k+1})$. Let $t(s)$ be r_s+1 if $Y(v_k^s, u(s), i_{k+1})$ is defined for all $k<r_s$; otherwise, let $t(s)$ be the least $k \leqq r_s$ for which $Y(v_{k-1}^s, u(s), i_k)$ is not defined. We define $v(s)$ and $Q(i, s)$ for all i:

$$v(s) = \begin{cases} w(u(s)) & \text{if } t(s) = 1, \\ (Y(v_{k-1}^s, u(s), i_k))_0 & \text{if } t(s) = k+1 > 1. \end{cases}$$

$$Q(i, s) = \begin{cases} Q(i, s-1) & \text{if } i < i_{t(s)}, \\ 0 & \text{if } i = i_{t(s)} \leqq s, \\ 1 & \text{if } i > i_{t(s)} \text{ or if } i > s. \end{cases}$$

Let $h(n)$ be the unique function associated with the sequence $\{u(0), u(1), u(2), \cdots\}$ of sequence numbers. It is clear from the definition of $u(s)$ that $h(n)$ is nonrecursive. To see that $h(n)$ has degree less than or equal to $0'$, observe that for each fixed $s>0$, $u(s)$ can be computed if the value of $v(s-1)$ and finitely many truth-values of $(Ey)T_1^1(\bar{0}(y), s, x, y)$ are known, $t(s)$ can be computed if the values of $u(s)$, $Q(0, s-1)$, $Q(1, s-1), \cdots, Q(s-1, s-1)$ and finitely many truth-values of $(Ey)T_1(e, x, y)$ are known, and both $v(s)$ and $Q(i, s)$ for all i can be computed if the values of $u(s)$, $t(s)$, $Q(0, s-1)$, $Q(1, s-1), \cdots, Q(s-1, s-1)$ are known.

We now show by induction on i that for each i there is an s^{**} such that $Q(i, s-1)=Q(i, s)$ for all $s \geqq s^{**}$. Suppose this is so for all $i<k$. Let s^* be such that $Q(i, s-1)=Q(i, s)$ for all $i<k$ and all $s \geqq s^*$. Suppose (for the sake of a reductio ad absurdum) that $s' \geqq s^*$, $Q(k, s'-1)=0$ and $Q(k, s')=1$. It follows from the definition of $Q(k, s')$ that $0 \leqq i_{t(s')} < k$, $Q(i_{t(s')}, s'-1)=1$ and $Q(i_{t(s')}, s')=0$. But this last is impossible because either $k=0$ or $s' \geqq s^*$. It must be the case that there is an s^{**} such that $Q(k, s'-1)=Q(k, s')$ for all $s' \geqq s^{**}$. For each i, let $s(i)$ be the least s such that $Q(i, s'-1)=Q(i, s')$ for all $s' \geqq s$. It can be shown that the function $s(i)$ is not recursive.

We define a contracting sequence of sets of sequence numbers. We set F_0 equal to the recursively enumerable set which has $V(q, u(s(0)), 0)$ as a Gödel number if $Q(0, s(0)-1)=1$, and equal to $\{s \mid \mathrm{Ext}(s, u(s(0)))\}$ otherwise. For each $j>0$, let f_{j-1} be a Gödel number of F_{j-1}. We set F_j equal to the recursively enumerable set which has $V(f_{j-1}, u(s(j)), j)$ as a Gödel number if $Q(j, s(j)-1)=1$, and equal to $\{s \mid \mathrm{Ext}(s, u(s(j))), s \in F_{j-1}\}$ otherwise.

Suppose that $\{e\}^h(n)$ is defined for all n. We claim that either $\{e\}^h(n)$ is recursive or $h(n)$ is recursive in $\{e\}^h(n)$. Suppose that $Q(e, s(e)-1)=0$, then $\{e\}^h(n)$ is recursive. This is so, because for each n, there is an $s \in F_e$ and a d such that $T_1^1(s, e, n, d)$, and because for each such s and d, $U(d)=\{e\}^h(n)$. Suppose that $Q(e, s(e)-1)=1$, then $h(n)$ is recursive in $\{e\}^h(n)$. This is so because there is only one function $w(n)$ associated with F_e such that $\{e\}^w(n)=\{e\}^h(n)$ for all n. To compute $h(n)$ from $\{e\}^h(n)$, we merely simultaneously enumerate F_e and the set of all deductions; whenever a choice has to be made between two sequence numbers, s_1 and s_2, of F_e, only one of which, let us say s_2, represents an initial segment of $h(n)$, there is nothing to fear because eventually some deduction will make clear that $(Ed, b)(T_1^1(s_1, e, b, (d)_0) \& U((d)_0) \neq \{e\}^h(b) \& T_1^1(s_2, e, b, (d)_1) \& U((d)_1) = \{e\}^h(b))$.

This completes the proof of Theorem 1 below. By making inessential changes Theorem 2 is proved.

THEOREM 1. *There exists a minimal degree less than* $0'$.

THEOREM 2. *For each degree c, there is a degree g greater than c and less than c′ such that* $c < b < g$ *for no degree b.*

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