A BASIS FOR NUMBER THEORY IN FINITE CLASSES

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A *set* is a class that is an element of some further classes. For many theories of classes, all classes are sets in this sense, for others not. Wang showed how to derive the theory of natural numbers from a theory of classes which assumed no sets with more than two elements; but his theory assumed infinite classes. Indeed all the usual ways of deriving the natural numbers from a theory of classes require infinite classes, because of the definition of "is a natural number" and the related law of mathematical induction. For, after somehow defining 0 and a successor function, it is customary to define the natural numbers as the common elements of all the classes (each patently infinite) that contain 0 and are closed with respect to successor. Then the justification of mathematical induction is that if the class of all numbers that have the desired property contains 0 and is closed with respect to successor, it contains all numbers by definition.

Such appeal to infinite classes is avoidable by this inversion: say rather that x is a natural number when 0 is a common element of all classes containing x and closed with respect to predecessor. Classical induction is now justifiable by exploiting the class of numbers up to an arbitrary x that lack the desired property. By this strategy the theory of natural numbers proves to be derivable within a theory of classes in which all classes are sets and all sets are finite.

The theory of classes used is pure, in the sense that it is expressible wholly in the notation of truth functions, quantification, and the epsilon of membership. In these terms, notations for the empty class and the unit class $\{x\}$ of x are contextually definable in well-known ways. Thereupon, following Zermelo, 0 is taken as the empty class and the successor of x is taken as $\{x\}$. The definition of "$x \leqq y$," readily formalized, works downward: x belongs to every class that contains y and, whenever it contains $\{z\}$, contains z. Then "x is a natural number" is rendered "$0 \leqq x$," which amounts to the version in the preceding paragraph. Ordered pairs, the notions of function and correlation, and the notation of singular description are introduced by contextual definition in the usual ways, and then $x+y$ is defined as the z such that the things $\leqq y$ are in correlation with the things w such that $x \leqq w \leqq z$. (The correlation itself exists as a finite class.) A multiple of x is next defined as any number u such that 0 belongs to every class that contains u and contains a number v whenever it contains $v+x$; and thereupon $x \cdot y$ is defined as the multiple

z of x such that the multiples of x that are $\leqq z$ are in correlation with the things $\leqq y$. (Separate treatment is accorded the case $x=0\neq y$.) The construction of power is analogous.

The set-theoretic axioms, added to the elementary logic of truth functions and quantification, are three: the usual extensionality axiom, the axiom that $\{x, y\}$ exists for all x and y, and the axiom that if y is a function applying to just the things $\leqq x$ then the range of y exists. The Peano axioms and the standard recursions for sum, product, and power all prove derivable, so that classical number theory is at hand.

The more usual way of stating existence axioms for finite classes is to say that 0 exists and, for each x and y, the union of x and $\{y\}$ exists. If for purposes of higher mathematics an axiom assuming the existence of an infinite class were ever added to these, each thing would forthwith become an element of various infinite classes. But no such consequence would issue from adding an infinite class to the present system. Thus the possibility is preserved, even if some infinite classes are eventually added for purposes beyond number theory, of maintaining something like the old distinction between sets and other classes; viz., a distinction between classes that are admissible as elements of infinite classes and classes that are not.

Bibliography

1. Hao Wang, *A new theory of element and number*, J. Symb. Logic vol. 13 (1948) pp. 129–137.

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