INEQUALITIES FOR FORMALLY POSITIVE INTEGRO-DIFFERENTIAL FORMS

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In 1954 N. Aronszajn [1] proved an inequality for formally positive integro-differential forms which has been found very interesting in itself and which has had a strong influence on subsequent progress in elliptic partial differential equations. We propose to extend this inequality in respect to the class of possible domains of integration and in respect to the kind of norms involved.

Let $\{P_j\}$ be a finite set of differential operators of order m with continuous coefficients on the closure $\bar{G}$ of a domain $G \subset R^n$. Suppose that the characteristic polynomials[2] $p_j(x, \xi)$ have no common real zero $\neq 0$ for $x \in \bar{G}$ and no common complex zero $\neq 0$ for $x \in \bar{G} - G$. Then an inequality of the form

$$(1) \qquad \sum_j \int_G |P_j u|^p dx + \int_G |u|^p dx \geq c \int_G |D^m u|^p dx, \qquad p > 1,$$

holds for all functions u of class C^m on $\bar{G}$ and all derivatives $D^m u$ of order m. The inequality is valid for a large class of bounded domains G—finite sums of those with boundary of Lipschitz graph type [2; 4]—including, for example, all with smooth boundary, all convex domains, and all finite sums of such.[3] With minor modifications it is also valid for quite a large class of unbounded domains.

The full details of the statement of the theorem, as well as the proof, will be given in another paper. Here we prefer to show the proof in a case which, though special, still contains the idea. We will suppose:

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[2] The p_j are the polynomials corresponding to the leading parts of the P_j. Thus, they are homogeneous polynomials (in ξ) of degree m. They all have the trivial zero $\xi = 0$.

[3] In Aronszajn's theorem $p = 2$ and there is the (rather inconvenient) restriction that G have a boundary of class C^1. In return, however, the condition on complex zeros is weakened. It is only required that there be none with imaginary part orthogonal to the boundary of G at x. Aronszajn's conditions are necessary as well as sufficient. It is not so clear how they should be re-formulated in the present case of (possibly) irregular boundaries. It should be noted, however, that when G is bounded with boundary of class C^1 and the leading parts of the P_j have constant coefficients our conditions are equivalent to his.

(i) There is an open cone C with vertex at 0 such that if $x \in G$ then $x+C \subset G$.[4]

(ii) The P_j are homogeneous and have constant coefficients.

We use the following notation: $x=(x_1, \cdots, x_n)$ is a point in R^n; $\xi=(\xi_1, \cdots, \xi_n)$ is an n-dimensional vector, real or complex; $\alpha=(\alpha_1, \cdots, \alpha_r)$ where α_i is an integer, $1 \leqq \alpha_i \leqq n$; $|\alpha|=r$; $\xi^\alpha = \xi_{\alpha_1}\xi_{\alpha_2} \cdots \xi_{\alpha_r}$; and D_α is the differential operator $\partial^r/\partial x_{\alpha_1} \cdots \partial x_{\alpha_r}$. Thus, P_j is the differential operator obtained by formal replacement in p_j of ξ^α by D_α.

PROOF. The p_j have no common complex zero other than 0. Therefore, by the Hilbert Nullstellensatz, if f is any polynomial which vanishes at 0, then some power of f is in the ideal generated by the p_j. Applying this to each of the polynomials $\xi_1, \xi_2, \cdots, \xi_n$ we get that if m' is sufficiently large then every homogeneous polynomial of degree m' is in the ideal generated by the p_j. In particular,

$$\xi^\beta = \sum_j a_{\beta j} p_j \quad for \quad |\beta| = m'$$

where the $a_{\beta j}$ are polynomials which plainly can be taken to be homogeneous of degree $m'-m$. Hence

$$D_\beta = \sum_j A_{\beta j} P_j \quad for \quad |\beta| = m' \tag{2}$$

if $A_{\beta j}$ is the homogeneous differential operator with characteristic polynomial $a_{\beta j}$.

Any function u of class $C^{m'}$ and compact support in $\overline{G}$ can be represented by its m'th derivatives (Sobolev [5], Calderón [3]): If ϕ is a function of class C^∞ on the sphere $S=\{x: |x|=1\}$ which has compact support in $S \cap C$ and which has integral over S equal to 1, then

$$u(x) = \frac{(-1)^{m'}}{(m'-1)!} \sum_{|\beta|=m'} \int D_\beta u(x+y) \frac{y^\beta}{|y|^n} \phi\left(\frac{y}{|y|}\right) dy.$$

Whether the integration is over C or over R^n is immaterial because of the vanishing of ϕ. In this formula we replace D_β by its expression in (2) and integrate by parts to transfer the operator $A_{\beta j}$ from u to the other factor. We get

[4] For example, G might be any open convex cone (infinite). However, (i) gives some idea of the domains which can be treated. When G is bounded, (1) is easily localized, and the proof can be made to cover the case when G has the *local* property required by (i).

$$(3)\quad \begin{aligned} u(x) &= \frac{(-1)^m}{(m'-1)!}\sum_j \int P_j u(x+y) \sum_{|\beta|=m'} A_{\beta j}\left\{\frac{y^\beta}{|y|^n}\phi\left(\frac{y}{|y|}\right)\right\} dy \\ &= \sum_j \int P_j u(x+y) K_j(y) dy \quad for \quad x \in G. \end{aligned}$$

The kernels

$$K_j(y) = \sum_{|\beta|=m'} A_{\beta j}\left\{\frac{y^\beta}{|y|^n}\phi\left(\frac{y}{|y|}\right)\right\}$$

are of class C^∞ in $R^n - \{0\}$, are positively homogeneous of degree $m-n$, and vanish outside a closed sub-cone of C.

Formula (3) is our basic formula which we believe will be useful not only in the question at hand but in many others.

In order to derive (1) (in the present situation) we follow Calderón [3]. On the right side of (3) replace $P_j u$ by f_j, its extension to $R^n - G$ by 0. Then the right side of (3) becomes a convolution which extends u to a function $\tilde{u}$ defined on the whole R^n. The derivatives $D_\alpha \tilde{u}$, $|\alpha| = m$, are expressible in terms of the f_j by means of singular integrals.[5] Hence, with any norm for which the singular integrals are bounded transformations (in particular the L^p norm) we have

$$\|D_\alpha \tilde{u}\| \leqq c \sum \|f_j\|,$$

which gives (1) and various other inequalities besides.[6]

Bibliography

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[5] Note that the mth derivatives of the kernels K_j have integral over S equal to 0, and see [4, Chapter III].

[6] In the case of certain norms (see [2]) it is necessary to take f_j to be some other extension of $P_j u$ than the one by 0, in order to obtain $\|f_j\| \leqq c'\|P_j u\|$.