THE KRULL-SCHMIDT THEOREM FOR INTEGRAL GROUP REPRESENTATIONS

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Let R_0 be the ring of algebraic integers in an algebraic number field K, let P be a prime ideal in R_0, and let R_P (or briefly R) denote the ring of P-integral elements of K. Choose $\pi \in R_0$ such that πR is the unique maximal ideal in R. Further let K^* be the P-adic completion of K, with ring of P-adic integers R^*. For a fixed finite group G, we understand by the term "R_0G-module" a left R_0G-module which as R_0-module is torsion-free and finitely-generated; analogous definitions hold for RG- and R^*G-modules.

Swan [**9**; **10**] has recently proved that the Krull-Schmidt theorem is valid for projective R^*G-modules. We show here the following main result, which is a consequence of some work of Maranda [**3**; **4**]:

THEOREM 1. *The Krull-Schmidt theorem holds for arbitrary R^*G-modules, that is, if $M_1, \cdots, M_r, N_1, \cdots, N_s$ are indecomposable R^*G-modules such that*

$$M_1 \dotplus \cdots \dotplus M_r \cong N_1 \dotplus \cdots \dotplus N_s \tag{1}$$

(the notation indicating external direct sums), then $r=s$, and after renumbering the $\{N_j\}$ if need be, $M_1 \cong N_1, \cdots, M_r \cong N_r$.

To prove this and some corollaries we make use of the following results of Maranda [**3**; **4**].

(i) Let M and N be R^*G-modules, and let e be the largest integer for which P^e divides the order of G. If $M \cong N$ then

$$M/\pi^d M \cong N/\pi^d N \qquad \text{as } (R^*/\pi^d R^*)G\text{-modules} \tag{2}$$

for all d.

Conversely if (2) holds for some $d > e$, then $M \cong N$. Furthermore, the same result holds for RG-modules.

(ii) Let M and N be RG-modules. Then $M \cong N$ if and only if $R^*M \cong R^*N$.

(iii) Let M be an R^*G-module. If M is decomposable, so is $M/\pi^d M$ for all d. Conversely if $M/\pi^d M$ is decomposable as $(R^*/\pi^d R^*)G$-module for some $d > 2e$, then M is also decomposable.

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Now fix $d=2e+1$, and let $\overline{M}=M/\pi^d M$, $\overline{R}^*=R^*/\pi^d R^*$, and so on. If (1) holds then we have

$$\overline{M}_1 \dotplus \cdots \dotplus \overline{M}_r \cong \overline{N}_1 \dotplus \cdots \dotplus \overline{N}_s$$

as $\overline{R}G$-modules, and each of the above summands is indecomposable by virtue of (iii). But $\overline{R}^*G$ is a ring with minimum condition, and so the Krull-Schmidt theorem is valid for $\overline{R}^*G$-modules (see [2]). Therefore $r=s$, and renumbering the $\{\overline{N}_j\}$ if need be, we have

$$\overline{M}_1 \cong \overline{N}_1, \cdots, \overline{M}_r \cong N_r.$$

The conclusion now follows by (i).

COROLLARY 1. *Let L, M, N be RG-modules such that $M \dotplus L \cong N \dotplus L$. Then $M \cong N$.*

COROLLARY 2. *Let $M^{(t)}$ denote the direct sum of t copies of M. If M, N are RG-modules such that $M^{(t)} \cong N^{(t)}$ for some t, then $M \cong N$.*

COROLLARY 3. *The Krull-Schmidt theorem holds for indecomposable RG-modules which remain indecomposable in passing to R^*. In particular, it is valid for absolutely irreducible RG-modules.*

The next corollary partially answers a question raised by Swan [10].

COROLLARY 4. *Let L, M, N be R_0G-modules such that $M \dotplus L \cong N \dotplus L$. Then for each P we have $R_P M \cong R_P N$. (If in particular M is absolutely irreducible, and R_0 has class number 1, then from Maranda [4] we may conclude that $M \cong N$.)*

It is still an open question as to whether the Krull-Schmidt theorem holds for RG-modules. That it fails for R_0G-modules already follows from [5], but the following approach is also instructive. Let N be an R_0G-submodule of the R_0G-module M, such that $KN=KM$, and define

$$\operatorname{ann}(M/N) = \{\alpha \in R_0 : \alpha M \subset N\}.$$

Then we have

THEOREM 2. *Let N_1, N_2 be submodules of the R_0G-module M such that $KN_1=KN_2=KM$, and suppose that*

$$\operatorname{ann}(M/N_1) + \operatorname{ann}(M/N_2) = R.$$

Then $N_1 \dotplus N_2 \cong M \dotplus (N_1 \cap N_2)$.

PROOF. Choose $\alpha_i \in \operatorname{ann}(M/N_i)$, $i=1, 2$, so that $\alpha_1+\alpha_2=1$. Then

$(n_1, n_2) \to (n_1+n_2, \alpha_2 n_1 - \alpha_1 n_2)$ gives the desired isomorphism.

In particular let M be absolutely irreducible, and let C denote an ideal in R_0. From [4] or [6] it follows that $M \cong CM$ if and only if C is principal. Hence if R_0 has class number >1, and if A, B are non-principal ideals of R_0 such that $A+B=R$, then we have from the above

$$AM \dotplus BM \cong M \dotplus ABM,$$

which shows that the Krull-Schmidt theorem does not hold.

Using a result of D. G. Higman's [1] (see also [6]) one can show that Theorems 1 and 2 are still valid when R_0G is replaced by an R_0-order in a separable K-algebra, and likewise with R or R^* in place of R_0.

Related problems are studied in [7; 8; 11].

Added in proof. The author has recently discovered that Theorem 1 has been proved previously by R. G. Swan [unpublished] and also by Borevich and Faddeyev [12], by a different approach. The corollaries and Theorem 2 are new, however.

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