ON THE CENTRAL LIMIT THEOREM IN R_k

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Communicated by J. L. Doob, March 6, 1961

Let $\xi_1, \xi_2, \cdots, \xi_n, \cdots$, be a sequence of independent and identically distributed random vectors in R_k with finite second order moments. Let $\eta_n = (\xi_1 + \cdots + \xi_n)n^{-1/2}$ and let $P_n(A) = P[\eta_n \in A]$. Let η denote a random vector in R_k which is normally distributed and whose moments of the first two orders are identical with those of ξ_1 and let $P(A) = P[\eta \in A]$. Then, by the central limit theorem in R_k, P_n weakly converges to P. A question that arises naturally here is an investigation of the error of approximation $P_n - P$. This problem has been thoroughly investigated in the case $k=1$ (cf. [**3**; **4**; **5**] and also the survey [**6**] where a complete set of references is given). For $k>1$, Bergström [**1**; **2**] obtained bounds on the error

$$\sup_{x \in R_k} | F_n(x) - \Phi(x) |$$

where F_n, Φ are the distribution functions of η_n and η respectively. Esseen [**5**] gave similar bounds for the error $|P_n(A) - P(A)|$, when A is a sphere with centre at the origin. The object of this study is to investigate the error $\Delta_n(A) = P_n(A) - P(A)$ for a very wide class of sets—namely the class of all convex subsets of R_k.

2. **Notation and preliminaries.** Let $\xi_1 = (\xi_1^{(1)}, \cdots, \xi_1^{(k)})$. We suppose that $E\xi_1^{(j)} = 0$ for $j = 1, 2, \cdots, k$, and that the variance covariance matrix of ξ_1, to be denoted by V, is nonsingular. We use the following notation for denoting the moments and cumulants of ξ_1:

$$\beta_s = \sum_{j=1}^{k} E | \xi_1^{(j)} |^s;$$

the cumulant of order $(s_1, s_2, \cdots, s_k)$ will be denoted by $\lambda_1^{s_1} \cdot \lambda_2^{s_2} \cdot \cdots \cdot \lambda_k^{s_k}$. Let $f(t)$ denote the characteristic function of ξ_1. Then the characteristic function of η_n is $[f(tn^{-1/2})]^n$. Let the polynomials $\tilde{P}_j(w)$ in the vector $w = (w_1, \cdots, w_k)$ be defined by the formal identity:

$$(1) \quad \exp\left\{ \sum_{j=3}^{\infty} \frac{1}{j!} (\lambda_1 w_1 + \cdots + \lambda_k w_k)^j n^{-(j-2)/2} \right\} = \sum_{j=0}^{\infty} n^{-j/2} \tilde{P}_j(w).$$

(Here the λ's represent the cumulants of ξ_1.) Let the functions $\tilde{P}_j(-\phi)$, $\tilde{P}_j(-\Phi)$ for $j = 0, 1, \cdots$, be defined as follows:

$$(2) \qquad \tilde{P}_j(-\phi)(x) = (2\pi)^{-k} \int \tilde{P}_j(it) \exp\left[-it' \cdot x - \frac{1}{2} t' V^{-1} t \right] dt$$

and

$$(3) \qquad \tilde{P}_j(-\Phi)(x) = \int_{[y_j \leq x_j;\, j=1,2,\cdots,k]} \tilde{P}_j(-\phi)(y) dy_1 \cdots dy_k.$$

A random vector ξ in R_k is said to be a lattice vector if there is a lattice $\mathcal{L}$ of points in R_k such that $P[\xi \in \mathcal{L}] = 1$. The lattice $\mathcal{L}$ is said to be minimal if there is no proper sublattice $\mathcal{L}_1$ of $\mathcal{L}$ such that $P[\xi \in \mathcal{L}_1] = 1$. If ξ_1 is a lattice vector we may always suppose without loss of generality that the minimal lattice in which ξ_1 is concentrated is $\mathcal{L}_0 = [a + m$; where a is some fixed vector and m is an arbitrary vector such that $m = (m_1, \cdots, m_k)$ where each m_j is an integer positive, negative or zero]. If ξ_1 is a lattice vector then we define

$$(4) \qquad p_n(z) = P[\xi_1 + \cdots + \xi_n = z] = P[\eta_n = zn^{-1/2}].$$

3. **Theorems.**

THEOREM 1. *Suppose that $\beta_4 < \infty$ and that the variance covariance matrix of ξ_1 is the identity matrix. Then*

$$(5) \qquad | P_n(A) - P(A) | \leq c(k) \beta_4^{3/2} (\log n)^{\alpha} n^{-1/2}$$

uniformly for all (measurable) convex subsets A of R_k. In (5), $\alpha = (k-1)/2(k+1)$ and $c(k)$ is a constant depending only on k.

THEOREM 2. *Suppose that the characteristic function of ξ_1 satisfies the condition* (C):

$$\limsup_{|t| \to \infty} | f(t) | < 1.$$

If $\beta_s < \infty$ $(s \geq 3)$ then

$$(6) \qquad P_n(A) = \sum_{j=0}^{s-3} n^{-j/2} \int_A d\tilde{P}_j(-\Phi) + O\{(\log n)^{(k-1)/2} n^{-(s-2)/2}\}$$

uniformly for all (measurable) convex subsets of R_k, where the functions $\tilde{P}_j(-\Phi)$ are defined by (3).

Now suppose that ξ_1 is a lattice vector concentrated in the minimal lattice $\mathcal{L}_0$. Let $S_1(u) = u - [u] + 1/2$ for all real numbers u. Then we have

THEOREM 3. *If $\beta_s < \infty$, then*

$$\sum_{z \in \mathfrak{L}_0} \left| p_n(z) - q_n^{(s)}(z) \right| = O\{(\log n)^{k/2} n^{-(s-2)/2}\}$$

where $p_n(z)$ *is defined by* (4) *and*

$$q_n^{(s)}(z) = \sum_{j=0}^{s-3} n^{-j/2} \tilde{P}_j(-\phi)(z).$$

THEOREM 4. *Suppose* $\beta_5 < \infty$. *Then*

$$P_n(A) = \int_A dQ(x) + n^{-1/2} \int_A d\tilde{P}_1(-\Phi) + O(n^{-1})$$

uniformly for all Borel sets A, *where*

$$Q(x) = \prod_{j=1}^{k} \left[1 - n^{-1/2} S_1(x_j n^{1/2} - na_j) \frac{\partial}{\partial x_j} \right] \Phi(x).$$

In particular

$$F_n(x) = \Phi(x) + n^{-1/2} \tilde{P}_1(-\Phi) - n^{-1/2} \sum_{j=1}^{k} S_1(x_j n^{1/2} - na_j) \frac{\partial \Phi}{\partial x_j} + O(n^{-1})$$

uniformly for x *in* R_k.

For Theorems 1 and 2 the method followed is a convolution method similar to the one employed by Esseen [5]. Theorem 3 is easily proved by standard techniques of Fourier analysis, and the transition from Theorem 3 to Theorem 4 is effected through a generalization of the classical Euler-Maclaurin sum formula to functions of several variables. The details of proofs will appear elsewhere.

The author is greatly indebted to R. R. Bahadur for encouragement and for valuable suggestions and discussions.

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