RESEARCH ANNOUNCEMENTS

The purpose of this department is to provide early announcement of significant new results, with some indications of proof. Although ordinarily a research announcement should be a brief summary of a paper to be published in full elsewhere, papers giving complete proofs of results of exceptional interest are also solicited.

ONTO INNER DERIVATIONS IN DIVISION RINGS

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Communicated by N. Jacobson, March 17, 1961

1. **Introduction.** Kaplansky [3] proposed the following problem: Does there exist a division ring Δ each element of which is a sum of additive commutators $ab-ba$? In [1] Harris gave a strongly affirmative solution to this problem by constructing division rings Δ in which each element $c=ab-ba$ for some $a, b\in\Delta$. Recently Meisters [4] has studied rings $R\neq(0)$ in which for any triple of elements $a, b, c\in R$ with $a\neq b$ there exist solutions of the equation $ax-xb=c$. He has shown that (1) R is a division ring in which every noncentral element induces an onto inner derivation and (2) if R is separable algebraic over its center, then R is commutative. Actually one can prove the more general result that in a division ring R of the preceding type all algebraic elements (over the center) are central. (Hence if R is noncommutative, each noncentral element $t\in R$ is transcendental over the center of R and induces an onto inner derivation.)

In view of the above work it seems natural to investigate the question of existence of division rings possessing onto inner derivations. We give a partial answer to this question which implies (in some heuristic sense) that Harris' examples (at least for char. $p>0$) are normative rather than pathological. More precisely we sketch a proof of the following theorem: For each division ring Δ of char. $p>0$ one can construct an extension division ring E with the property that there exists an element $t\in E$ (lying in the centralizer of Δ) whose associated inner derivation D_t is an onto map: $D_t(E)=E$.

2. **Preliminaries.** We shall make consistent use of the following facts: (1) Any noncommutative ring R with an identity having the common right multiple property has a right quotient ring $Q(R)$, i.e., every element of $Q(R)$ has the form ab^{-1}, $a, b\in R$, b regular, and all regular elements of R are invertible in $Q(R)$. (2) If Δ is a division ring and D a derivation of Δ into itself, then $\Delta[x; D]$, the ring of differential polynomials over Δ in the indeterminate x, has the com-

mon right multiple property; thus by (1), $\Delta[x; D]$ has a quotient division ring $Q(\Delta[x; D])$, since all nonzero elements in $\Delta[x; D]$ are regular. (3) If R is a ring with quotient ring $Q(R)$ and D is a derivation of R into an extension ring S of $Q(R)$, then D can be uniquely extended to a derivation of $Q(R)$ into S by defining, for $ab^{-1} \in Q(R)$, $D(ab^{-1}) = D(a)b^{-1} - (ab^{-1})(D(b)b^{-1})$.

A proof of (1) may be found in [**2**, p. 118]; (2) was established in [**5**]; and (3) is a fairly straightforward exercise in computation. Finally note that in rings of char. $p>0$ all p^nth powers $(n \geqq 0)$ of a derivation are again derivations.

3. **The construction.** Let Δ_0 be the quotient division ring of the polynomial ring $\Delta[t]$ (Δ a division ring of char. $p>0$) where t is a commuting indeterminate over Δ. Set $x_0 = 1$ and let D_0 be the unique extension of ordinary differentiation in $\Delta[t]$ to Δ_0 so that D_0 is a derivation of Δ_0 into itself. Choose an indeterminate x_1 over Δ_0 and form the quotient division ring $\Delta_1 = Q(\Delta_0[x_1; D_0])$. Noting that $D_t(x_1) = x_0$ and $D_0(x_0) = 0$, we see that we have verified the case $n=0$ of the proposition: Given $\Delta_0 = Q(\Delta[t])$ there exists a nested sequence of division rings Δ_n, a set of derivations $D_n: \Delta_n \to \Delta_n$, and elements $x_n \in \Delta_n$ satisfying

$$\Delta_{n+1} = Q(\Delta_n[x_{n+1}; D_n]), \tag{1}$$

$$D_t(x_{n+1}) = x_n, \tag{2}$$

$$D_n(t) = x_n, \qquad D_n(x_i) = 0, \qquad i = 0, \cdots, n; n \geqq 0. \tag{3}$$

To prove this proposition we proceed by induction. Suppose the truth of the proposition for $n = 0, \cdots, s$. Then we have constructed Δ_n, D_n, x_n, for $n = 0, \cdots, s$, satisfying the above conditions. Choose an indeterminate x_{s+1} over Δ_s and let $\Delta_{s+1} = Q(\Delta_s[x_{s+1}; D_s])$. We must construct a derivation $D_{s+1}: \Delta_{s+1} \to \Delta_{s+1}$ satisfying $D_{s+1}(t) = x_{s+1}$, $D_{s+1}(x_i) = 0$ $(i = 0, \cdots, s+1)$, and $D_t(x_{s+1}) = x_s$. We do this by defining D_{s+1} on Δ_0 and extending it to each successive Δ_i $(i = 1, \cdots, s+1)$ as follows. Suppose D_{s+1} has been defined on Δ_l, $0 \leqq l < s+1$; then to define it on Δ_{l+1} we need only check that it can be extended to $\Delta_l[x_{l+1}; D_l]$. Now if $\sum a_i x_{l+1}^i$, $a_i \in \Delta_l$, is a typical element of this ring we set $D_{s+1}(\sum a_i x_{l+1}^i) = \sum D_{s+1}(a_i) x_{l+1}^i$. Since the map $D_{s+1}D_l - D_l D_{s+1}$ is zero on Δ_l, one verifies that D_{s+1} as defined is a derivation on Δ_{l+1}. Thus if D_{s+1} can be constructed on Δ_0 we shall be done. Let $a \in \Delta[t]$. Define

$$D_{s+1}(a) = \sum_{i=0}^{s+1} D_0^{i+1}(a)/(i+1)! x_{s+1-i} \pmod{p}.$$

This makes sense since the coefficients of $D_0^{i+1}(a)$ are divisible by $(i+1)!$. Observing that $x_l a = \sum_{i=0}^{l} D_0^i(a)/i! x_{l-i}$ (mod p), $l=0, \cdots, s+1$, one verifies that D_{s+1} is a derivation on $\Delta[t]$ and hence on Δ_0. By what we have said previously it has an extension to Δ_{s+1} and clearly satisfies all requisite properties.

Next let $E=\bigcup_{n=0}^{\infty}\Delta_n$. Since $D_t(x_n)=x_{n-1}$ we get $D_t^{n+1}(x_n)=0$ and therefore there exists a least integer $l\geqq 0$ for which $D_{t^{p^l}}(x_n)=0$. It is immediate that $D_{t^{p^l}}(\Delta_n)=0$, so Δ_n is contained in the centralizer of t^{p^l}. But $D_{t^{p^l}}(x_{p^l})=1$, hence if a is in the centralizer of t^{p^l}: $x^{p^l}at^{p^l}-t^{p^l}x^{p^l}a=a$. It follows, since $x^{p^l}a$ is in $\Delta_{p^{l+1}}$, that $D_{t^p}(\Delta_{p^{l+1}})\supseteq\Delta_n$. But $D_t(\Delta_{p^{l+1}})\supseteq D_{t^p}(\Delta_{p^{l+1}})\supseteq\Delta_n$. As n was arbitrary, $D_t(E)=E$.

References

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