A CONTINUOUS FUNCTION WITH TWO CRITICAL POINTS

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A real C^s-function $f: X \to \mathbf{R}$ on an n-dimensional C^s-manifold with $s \geqq 0$, is called *C^s-nondegenerate C^s-ordinary* at a point $p \in X$, in case a system of n C^s-coordinates (C^s-functions) $\phi_1, \cdots, \phi_n$ exists, which defines a C^s-diffeomorphism κ of some neighborhood $V(p)$ of p into $\mathbf{R}^n$, and such that for some constant $\lambda_p > 0$

$$(1)\phi_i(p) = 0,\ i = 1, \cdots, n;\ \phi_n(q) = \lambda_p\{f(q) - f(p)\}$$

for $q \in V(p) \subset X$.

If C^s-coordinates and $\lambda_p > 0$ exist such that

$$(2)\qquad \begin{aligned} \phi_i(p) &= 0, \qquad i = 1, \cdots, n; \\ -\sum_1^r \phi_i^2(q) + \sum_{r+1}^n \phi_j^2(q) &= \lambda_p\{f(q) - f(p)\} \end{aligned}$$

then the function is called *C^s-critical of index r and C^s-nondegenerate at p.*

A function which is C^s-nondegenerate at every point $p \in X$ is called a *C^s-nondegenerate function.*

We will restrict our considerations to the topological case $s = 0$ of continuous functions on topological manifolds and we will omit C^0 from the notation in the sequel. By *function we will mean continuous function*, etc.

A compact manifold without boundary is called a *closed* manifold. A nondegenerate function on a closed manifold has at least one critical point p_1 of index n and one critical point p_0 of index 0, corresponding respectively with the maximum and the minimum of the function. We prove the

THEOREM. *If X is a closed n-dimensional manifold and $f: X \to \mathbf{R}$ a continuous nondegenerate function with exactly two critical points, then X is homeomorphic to the n-sphere S^n.*[2]

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[2] Reeb [2] proved the corresponding theorem for the differentiable case. Morse [1] proved that X is a homotopy-sphere, and he also has a proof of the theorem we present (unpublished as yet).

PROOF. A. *The local droppings* T_p. We place ourselves in the assumptions of the theorem and we call the function f "height." We consider a coordinate system for every point $p \in X$, obeying (1) or (2), but for which moreover the image $\kappa_p(V(p)) \subset \mathbf{R}^n$ is the open n-ball

$$r < 5, \tag{3}$$

where the "*polar coordinates*" r (radius) and ω (unit vector) are defined by

$$r = \left(\sum_j^n \phi_j^2 \right)^{1/2}, \qquad \omega = (\phi_1/r, \phi_2/r, \cdots, \phi_n/r). \tag{4}$$

For any such coordinate system κ: $V(p) \to \mathbf{R}^n$ we also define the open set

$$U_t(p) = \{ q \mid q \in V(p) \subset X, r(q) < t \}. \tag{5}$$

Next we define a homeomorphism T_p for every $p \in X$. If p is an *ordinary point* then we proceed as follows:

Let $h(t)$ be a real C^∞-function with the properties

$$h(t) \begin{cases} = 0, & |t| \geqq 4, \\ > 0, & |t| < 4, \\ = h(0), & |t| \leqq 1, \end{cases} \qquad |h'(t)| < 1/2, \quad \text{any } t. \tag{6}$$

The homeomorphism T_p is given by:

$$\left.\begin{aligned} \phi_i(T_p(q)) &= \phi_i(q), \quad i = 1, \cdots, n-1 \\ \phi_n(T_p(q)) &= \phi_n(q) - h(r(q)) \end{aligned}\right\} \quad q \in V(p), \qquad T_p(q) = q, \quad q \notin V(p). \tag{7}$$

As the Jacobian of the corresponding C^∞-transformation of the coordinates for $q \in V(p)$ does not vanish, and $T(q) = q$ for $q \notin U_4(p)$, it follows that T_p is a global homeomorphism of X. Observe that the continuous function

$$q \to f(T_p(q)) - f(q) : X \to \mathbf{R}$$

takes the value zero for $q \notin U_4(p)$ and is negative for $q \in U_4(p)$. It takes a negative maximal value on the set $\overline{U_3(p)}$, the closure in X of $U_3(p)$. Under T_p no point is mapped into a higher level of f, and every point of $U_4(p)$ is mapped into a lower level.

If p is a *critical point of index n* we use a real C^∞-function $k(t)$ with the properties

$$k(t)\begin{cases} = t, & \text{for } t \geqq 4, \\ = 2t, & 0 \leqq t \leqq 1, \\ > t, & 0 \leqq t < 4, \end{cases} \tag{8}$$
$$k'(t) > 0, \qquad t \geqq 0.$$

The homeomorphism T_p is now defined in terms of polar coordinates (4) by:

$$\left.\begin{aligned} \omega(T_p(q)) &= \omega(q) \\ r(T_p(q)) &= k(r(q)) \end{aligned}\right\} \text{ for } q \in V(p), \tag{9}$$
$$T_p(q) = q \qquad \text{for } q \notin V(p).$$

The restriction of T_p to $U_1(p)$ is represented by a geometrical multiplication with factor 2 in coordinate space.

The point p and every point $q \notin U_4(p)$ is invariant under T_p. Every other point in X is mapped into a lower level.

In the case of *critical point of index zero* we use the function k^{-1}, the inverse of k, and proceed analogously.

B. *The global dropping T.* Under the given assumptions there is a critical point p_1 of index n (maximum), a critical point p_0 of index 0 (minimum), and no other critical point. Choose a finite number of coordinate systems κ_{p_i} and homeomorphisms T_{p_i}, $i=0, \cdots, L$, of the kinds mentioned above, such that:

$$\bigcup_{i=0}^{L} U_3(p_i) = X \tag{10}$$

but

$$\bigcup_{i=2}^{L} U_4(p_i) \cap [U_2(p_0) \cup U_2(p_1)] = \emptyset \text{ (void)}. \tag{10}$$

(Compare the use of a partition of unity.)

Let

$$T = T_{p_L} T_{p_{L-1}} \cdots T_{p_2} T_{p_1} T_{p_0}. \tag{11}$$

Then $T: X \rightarrow X$ is a global homeomorphism with exactly two invariant points, namely p_0 and p_1, which maps every other point into a lower level:

$$T(p_0) = p_0; \quad T(p_1) = p_1; \quad f(T(q)) \leqq f(q) \quad \text{for } q \in X - p_0 - p_1. \tag{12}$$

As the set $W_\epsilon = X - U_2(p_1) - U_\epsilon(p_0)$ for $0<\epsilon<1$, is compact, the non-negative function

$$f(q) - f(T(q))$$

has a minimal value for $q \in W_\epsilon$ and this minimal value is positive. Call it $\delta_\epsilon > 0$ and let N_ϵ be an integer such that

$$N_\epsilon \delta_\epsilon > f(p_1) - f(p_0). \tag{13}$$

If we apply powers with consecutive exponents of the homeomorphism T, to any point $q \in W_\epsilon$, then for some exponent $N \leqq N_\epsilon$ we will find

$$T^N(q) \in U_\epsilon(p_0)$$

because with each new application of T to the result obtained in the last step, we obtain a new point which is at a level at least δ_ϵ lower, and after N_ϵ steps the point would have dropped totally more than the total range of the function f over X. On the other hand, once the resulting point is in $U_\epsilon(p_0)$ any further application of T will give a new point also in $U_\epsilon(p_0)$, because T acts in $U_\epsilon(p_0)$ as a geometrical multiplication with factor 1/2. Consequently

$$T^{N_\epsilon}(X - U_2(p_1)) \subset U_\epsilon(p_0) \tag{14}$$

and taking complements

$$T^{N_\epsilon}(U_2(p_1)) \supset X - U_\epsilon(p_0). \tag{14c}$$

Thus X is covered by two discs:

$$T^{N_\epsilon}(U_2(p_1)) \cup U_\epsilon(p_0) = X \tag{15}$$

and our theorem can be considered as a consequence of a theorem of Morton Brown. However, we like to present a complete explicit proof:

C. *The homeomorphism* $X \to S^n$.

As (14) holds for any $0<\epsilon<1$, it follows that for any $q \neq p_0$ there exists a smallest number N_q such that

$$T^{N'} U_2(p_1) \ni q \qquad \text{for } N' \geqq N_q$$

or

$$T^{-N'}(q) \in U_2(p_1). \tag{16}$$

Let $\kappa_1 \colon U_2(p_1) \to \mathbf{R}^n$ be the restriction of the coordinate system at the critical point p_1 to the open set $U_2(p_1)$. Observe that for any $q \in U_2(p_1)$:

$$2^k \cdot \kappa_1[T^{-k}(q)] = \kappa_1(q), \qquad k \geqq 0. \tag{17}$$

If $N' \geqq N = N_q$ then in view of (17) we have in the vector space $\mathbf{R}^n$:

$$2^{N'}\kappa_1[T^{-N'}(q) = 2^N \cdot 2^{N'-N}k_1(T^{-N'+N}T^{-N}q) = 2^N\kappa_1(T^{-N}q).$$

Hence there exists a mapping κ: $(X - p_0) \rightarrow \mathbf{R}^n$ well defined by:

$$\kappa(q) = 2^{N'}\kappa_1(T^{-N'}q), \qquad N' \geqq N_q. \tag{18}$$

κ is clearly locally a homeomorphism. κ is onto the set $\bigcup_{j=0}^{\infty} 2^j\kappa_1(U_2(p_1)) = \mathbf{R}^n$. If q_1 and $q_2 \neq q_1$ are both different from p_0 then, for $N' \geqq N_{q_1} + N_{q_2}$,

$$T^{-N'}(q_1) \neq T^{-N'}(q_2)$$

and consequently $\kappa(q_1) \neq \kappa(q_2)$. So κ: $(X - p_0) \rightarrow \mathbf{R}^n$ is a homeomorphism and X is homeomorphic to the one point compactification of $\mathbf{R}^n$, that is S^n.

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