A NEW CLASS OF SPECTRAL OPERATORS[1]

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Communicated by W. S. Massey, October 14, 1960

Let X be an ordered (=partially ordered) complex Banach space (cf. [3, §6]). The positive cone $K=\{x: x\geqq 0\}$ in X is normal if there exists $\gamma>0$ such that $\|x+y\|\geqq\gamma\|y\|$ for all $x, y\in K$. We say that a complex B-algebra A (with unit e) is *ordered* if the underlying B-space is ordered with K closed and normal, and if K, in addition, has these properties: (i) $e\in K$; (ii) $a\in K$, $b\in K$ and $ab=ba$ imply $ab\in K$ (cf. [3, §11]). We shall write, as usual, $x\leqq y$ (or $y\geqq x$) for $y-x\in K$, and $[x, y]=\{z: x\leqq z\leqq y\}$. The term "semi-complete" stands for "sequentially complete." For any $a\in A$, $\sigma(a)$ denotes the spectrum of a. A function μ from the Borel sets of the real line R into A is a *Borel measure* if μ is countably additive, i.e., if $\mu(\bigcup_1^\infty \delta_n)=\sum_1^\infty \mu(\delta_n)$ converges in A for an arbitrary sequence $\{\delta_n\}$ of mutually disjoint Borel sets.

THEOREM 1. *Let A be an ordered B-algebra, such that $[0, e]$ is weakly semi-complete. Let $c_1e\leqq a\leqq c_2e$ where $c_1, c_2\in R$. Then $\sigma(a)\subset[c_1, c_2]$, and there exists an A-valued Borel measure μ such that*

$$a^n=\int_{\sigma(a)} t^n d\mu, \qquad (n=0, 1, 2, \cdots).$$

Moreover, μ is a homomorphism of the Boolean σ-algebra of real Borel sets onto a Boolean σ-algebra of idempotents contained in $[0, e]$, and

$$f\rightarrow\int_{\sigma(a)} f d\mu$$

is an order preserving homomorphism of the algebra of bounded Borel functions on $\sigma(a)$, into A.

If A is a (Banach) algebra of bounded operators on a B-space X, then an $a\in A$ satisfying the assertions of Theorem 1 is a (scalar type) spectral operator in the sense of Dunford [1]. We obtain from Theorem 1:

THEOREM 2. *Let A be an ordered B-algebra of operators on a weakly semi-complete B-space X. Then every operator a which is contained in the real linear hull of $[0, e]$ is a scalar type spectral operator, $a=\int\lambda d\mu$, with real spectrum $\sigma(a)$, and μ is a spectral measure with values in $[0, e]$.*

[1] Research sponsored by the Office of Ordnance Research, U. S. Army.

COROLLARY. *Let $a \in A$. If the convex cone (of vertex 0) spanned by the set $\{a^m(e-a)^n : m, n=0, 1, 2, \cdots\}$ is normal, then a is a scalar type spectral operator with real spectrum.*

Let X be a complex Hilbert space of arbitrary dimension; the algebra A of bounded operators on X is ordered with respect to the familiar positivity notion for Hermitian elements of A; since every Hermitian operator is in the real linear hull of $[0, e]$, the spectral theorem for bounded Hermitian operators is a special case of Theorem 2. Theorem 2 can be extended to elements $a+bi$ where $ab=ba$ and a, b are both in the real linear hull of $[0, e]$. Thus the spectral theorem for (bounded) normal operators in Hilbert space is a consequence of Theorem 2. If A is an algebra of operators on an arbitrary Banach space, the analog of the cone of positive Hermitian operators is the cone K_s spanned by finite sums of squares; if K_s is normal and weakly semi-complete, all $a \in A$ are scalar type spectral operators with real spectrum. K_s has been considered, e.g., in [2]. Another general example can be obtained as follows.

THEOREM 3. *Let X be an ordered Banach space whose positive cone K is normal, weakly semi-complete, and generating.*[2] *Then the algebra A of bounded operators on X is an ordered B-algebra with positive cone $\{a \in A : aK \subset K\}$, and every element in the real linear hull of $[0, e]$ is a scalar type spectral operator such that $\sigma(a) \subset [c_1, c_2]$ if $c_1 e \leqq a \leqq c_2 e$ $(c_1, c_2 \in R)$.*

It follows from this theorem that there are nontrivial scalar type spectral operators on every weakly semi-complete Banach space. Proofs of the announced results and further results concerning compact operators and unbounded operators will be published elsewhere.

REFERENCES

1. N. Dunford, *A survey of the theory of spectral operators*, Bull. Amer. Math. Soc. vol. 64 (1958) pp. 217–274.

2. J. L. Kelley and R. L. Vaught, *The positive cone in Banach algebras*, Trans. Amer. Math. Soc. vol. 74 (1953) pp. 44–55.

3. H. Schaefer, *Halbgeordnete lokalkonvexe Vektorräume*. II, Math. Ann. vol. 138 (1959) pp. 259–286.

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[2] I.e., $X=K-K$.