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SUMMABILITY (L) OF FOURIER SERIES

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1. In a recent paper, Borwein [1] has constructed a new method of summability for an infinite sequence (S_n). He defined a sequence (S_n) to be summable by the logarithmic method of summability or summable (L) to the sum s if, for x in the interval $(0,1)$

$$\lim_{x\to 1-0} \frac{1}{|\log(1-x)|} \sum_{n=1}^{\infty} \frac{S_n}{n} x^n = s,$$

which is written simply as $S_n \to s(L)$. Concerning this kind of summability, Borwein has established a number of fundamental facts. For instance, he showed that $(L)\supset(A,\lambda)$.[1] Thus, we have the following full inclusive relation:

$$(L) \supset (A, \lambda) \supseteq (A) \supset (C, r),$$

for any $r>-1$, where (A) is the ordinary Abel's summability and (C, r) is the Cesàro summability of order r.

In this note, the author intends to apply this new method of summability to the Fourier series of $f(x)$ in order to obtain a corresponding summability criterion for it.

[1] A sequence (S_n) is said to be summable (A, λ) to the sum s if $(1-x)\sum S_n^\lambda x_n \to s$ as $x\to 1-0$, where S_n^λ is the nth Cesàro mean of order λ of (S_n) [1, p. 212 and §3, Theorem 3].

2. Suppose that $f(x)$ is a Lebesgue integrable function, periodic with period 2π. Let

$$f(x) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

be its Fourier series. Fixing x_0, we write

$$\phi(t) = \phi_{x_0}(t) = \frac{1}{2} \{f(x_0 + t) + f(x_0 - t) - 2s\}.$$

First, we derive the following fundamental theorem concerning the kernel of the summability (L) for Fourier series.

THEOREM 1. *The necessary and sufficient condition for the Fourier series of $f(x)$ to be summable (L) to the sum s at the point x_0 is that*

$$\int_0^{\pi} \frac{\phi(t)}{t} \tan^{-1} \frac{x \sin t}{1 - x \cos t} dt = o(|\log(1 - x)|)$$

as $x \to 1 - 0$.

Let

$$S_n(x_0) = \frac{1}{2} a_0 + \sum_{\nu=1}^{n} (a_\nu \cos \nu x_0 + b_\nu \sin \nu x_0)$$

be the nth partial sum of the Fourier series of $f(x)$ at x_0. Then, we have

$$S_n(x_0) - s = \frac{1}{\pi} \int_0^{\pi} \phi(t) \frac{\sin nt}{t} dt + o(1).$$

Thus,

$$\begin{aligned} &\sum_{n=1}^{\infty} \frac{1}{n} \{S_n(x_0) - s\} x^n \\ &= \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{x^n}{n} \int_0^{\pi} \phi(t) \frac{\sin nt}{t} dt + o\left(\sum_{n=1}^{\infty} \frac{x^n}{n}\right) \\ &= \frac{1}{\pi} \int_0^{\pi} \frac{\phi(t)}{t} \left(\sum_{n=1}^{\infty} \frac{\sin nt}{n} x^n\right) dt + o(|\log(1 - x)|) \\ &= \frac{1}{\pi} \int_0^{\pi} \frac{\phi(t)}{t} \tan^{-1} \frac{x \sin t}{1 - x \cos t} dt + o(|\log(1 - x)|). \end{aligned}$$

Now,

$$\sum_{n=1}^{\infty} \frac{1}{n}\{S_n(x_0) - s\}x^n = \sum_{n=1}^{\infty} \frac{S_n(x_0)}{n} x^n - s\,|\log(1-x)|$$
$$= L(x) - s\,|\log(1-x)|.$$

Hence, the sequence $(S_n(x_0))$ is summable (L) to s if and only if

$$\int_0^{\pi} \frac{\phi(t)}{t} \tan^{-1} \frac{x \sin t}{1 - x\cos t}\, dt = o(|\log(1-x)|)$$

as $x \to 1-0$. This establishes the theorem.

3. Next, we derive a summability criterion of (L) summability for the Fourier series of $f(x)$ at x_0 as follows.

THEOREM 2. *If*

(i) $$\int_0^t |\phi(u)|\, du = o(t\,|\log t|), \qquad (t \to +0),$$

(ii) $$\int_t^{\delta} (|\phi(u)|/u)\, du = o(|\log t|),$$

as $t \to +0$ *for any arbitrary* $0 < \delta < \pi$, *then the Fourier series of* $f(x)$ *is summable* (L) *to* s *at* x_0.

For, if we write

$$\int_0^{\pi} \frac{\phi(t)}{t} \tan^{-1} \frac{x \sin t}{1 - x\cos t}\, dt = \int_0^{1-x} + \int_{1-x}^{\delta} + \int_{\delta}^{\pi}$$
$$= J_1(x) + J_2(x) + J_3(x),$$

say. Then, since

$$\lim_{t \to +0} \frac{1}{t} \tan^{-1} \frac{x \sin t}{1 - x\cos t} = \frac{x}{1-x},$$

we can choose x_0 sufficiently near 1 such that

$$|J_1(x)| < \frac{2x}{1-x} \int_0^{1-x} |\phi|\, dt$$

for $0 < x_0 < x < 1$. It follows that $J_1(x) = o(|\log(1-x)|)$ as $x \to 1-0$ by (i). Considering that

$$\left|\tan^{-1} \frac{x \sin t}{1 - x\cos t}\right| < \frac{\pi}{2}$$

uniformly for $0 \leqq x < 1$ and $0 < t \leqq \pi$, we find

$$|J_2(x)| < \frac{\pi}{2} \int_{1-x}^{\delta} \frac{|\phi|}{t} dt = o(|\log(1-x)|)$$

as $x \to 1-0$ by (ii). Last, we have

$$\begin{aligned} |J_3(x)| &= \left| \int_{\delta}^{\pi} \frac{\phi}{t} \tan^{-1} \frac{x \sin t}{1 - x \cos t} dt \right| \\ &\leqq \frac{1}{\delta} \int_{\delta}^{\pi} |\phi| \left| \tan^{-1} \frac{x \sin t}{1 - x \cos t} \right| dt \\ &< \frac{\pi}{2\delta} \int_{0}^{\pi} |\phi| \, dt \\ &= O(1) \\ &= o(|\log(1-x)|) \end{aligned}$$

as $x \to 1-0$. This proves Theorem 2.

4. Accordingly, from the estimation of $J_3(x)$ in the proof of the above theorem, we get the following almost self-evident

THEOREM 3. *The (L) summability of the Fourier series of $f(x)$ at x_0 is a local property of $f(x)$ near x_0. I.e.,*

$$L(x) = \frac{1}{\pi} \int_{0}^{\delta} \frac{\phi(t)}{t} \tan^{-1} \frac{x \sin t}{1 - x \cos t} dt + o(|\log(1-x)|)$$

for any arbitrary $0 < \delta < \pi$ as $x \to 1-0$.

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