FOURIER EXPANSIONS OF ARITHMETICAL FUNCTIONS

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Communicated by Gerald B. Huff, October 5, 1960

Let $f(n)$, $g(n)$ denote complex-valued arithmetical functions with

$$f(n) = \sum_{d|n} g(d). \tag{1}$$

Wintner has proved [3, §33] that if

$$\sum_{n=1}^{\infty} \frac{g(n)}{n} \tag{2}$$

converges absolutely, then $f(n)$ is almost periodic (B) with the absolutely convergent Fourier expansion,

$$f(n) \sim \sum_{r=1}^{\infty} a_r c(n, r), \qquad a_r = \sum_{n=1;\, r|n}^{\infty} \frac{g(n)}{n}, \tag{3}$$

$c(n, r)$ denoting Ramanujan's trigonometric sum. In addition, Wintner showed [3, §35] that if

$$\sum_{n=1}^{\infty} \frac{\tau(n) \mid g(n) \mid}{n} \tag{4}$$

is convergent, where $\tau(n)$ denotes the number of divisors of n, then $f(n)$ is represented for all n by its Fourier series,

$$f(n) = \sum_{r=1}^{\infty} a_r c(n, r), \qquad a_r = \sum_{n=1;\, r|n}^{\infty} \frac{g(n)}{n}. \tag{5}$$

In this announcement we point out that for certain important classes of multiplicative functions $f(n)$ for which (2) is absolutely convergent (including many such examples that are familiar), the convergence of (4) is not needed to ensure the validity of (5). In fact, we have the following result.

THEOREM I. *Suppose that $f(n)$ is multiplicative and that* (2) *converges absolutely. In case either*

(i) $g(n)$ *is completely multiplicative,*

or in case

(iia) $g(n) = 0$ *when n is not square-free, and*

(iib) $g(p) \neq -p$ *for all primes p,*

then (5) *holds with the convergence absolute.*

An analogue of this theorem for functions of finite abelian groups is proved in [2, Theorems 6.1(a) and 7.3]. With appropriate changes in notation and terminology the proofs carry over to the case of the integers as expressed by Theorem I.

The above result can be reformulated in the following simple manner. Let $\sum'$ indicate summations restricted to square-free integers and place

$$h(n) = \sum_{d|n} \frac{g(d)}{d}. \tag{6}$$

THEOREM II. *Suppose that* $f(n)$ *is multiplicative and that* (2) *converges absolutely with sum* α. *Then if* (i) *is satisfied,*

$$f(n) = \alpha \sum_{r=1}^{\infty} \left(\frac{g(r)}{r}\right) c(n, r); \tag{7}$$

on the other hand, if (iia) *and* (iib) *are satisfied,*

$$f(n) = \alpha \sum_{r=1}^{\infty}{}' \left(\frac{g(r)}{rh(r)}\right) c(n, r), \tag{8}$$

the convergence of both (7) *and* (8) *being absolute.*

PROOF. Let $g(n)$ satisfy (iia) and (iib). The analogue for integers of [2, Lemma 7.2] asserts that $h(n) \neq 0$ for square-free n, and that

$$\sum_{n=1;\,(n,r)=1}^{\infty} \frac{g(n)}{n} = \frac{\alpha}{h(r)}, \qquad r \text{ square-free}. \tag{9}$$

By (1), $g(n)$ is multiplicative, and hence from Theorem I, (5) is valid with

$$a_r = \sum_{n=1;\, r|n}^{\infty} \frac{g(n)}{n} = \sum_{a=1}^{\infty} \frac{g(ra)}{ra} = \frac{g(r)}{r} \sum_{a=1;\,(a,r)=1}^{\infty} \frac{g(a)}{a} = \frac{\alpha g(r)}{rh(r)}$$

for square-free r, by virtue of (iia) and (9). Evidently, by (iia) and (5), $a_r = 0$ if r is not square-free. This proves (8). The proof of (7) is similar but simpler, and is therefore omitted.

Let us now consider some special cases. Let $\sigma(n)$ denote the sum of the divisors of n, $\phi(n)$ the Euler ϕ-function, $\psi(n)$ the Dedekind ψ-function [3, p. 43], and $\Phi(n)$ the "square-totient" [1, §6], denoting the number of a (mod n) such that (a, n) is a square. Corresponding to the cases, $f(n) = \sigma(n)/n$, $\phi(n)/n$, $\psi(n)/n$, and $\Phi(n)/n$, we have $g(n) = 1/n$, $\mu(n)/n$, $\mu^2(n)/n$, and $\lambda(n)/n$, respectively, where $\mu(n)$ denotes the Möbius function and $\lambda(n)$ Liouville's function. Each of the functions $f(n)$ is multiplicative with the series (2) absolutely convergent. In the cases $f(n)=\sigma(n)/n$ and $\Phi(n)/n$, the condition (i) is satisfied, with $\alpha=\pi^2/6$ and $\pi^2/15$ in the two cases, respectively. On the other hand, (iia) and (iib) are satisfied in the cases $f(n)=\phi(n)/n$ and $\psi(n)/n$, in which cases, $\alpha=6/\pi^2$ and $15/\pi^2$, respectively. Moreover, if $J_r(n)$ denotes the Jordan totient of order r, then $h(n)=J_2(n)/n^2$ when $f(n)=\phi(n)/n$, and $h(n)=J_4(n)/n^2J_2(n)$, when $f(n)=\psi(n)/n$. Hence, by Theorem II, we have the following absolutely convergent expansions:

$$\frac{\sigma(n)}{n}=\frac{\pi^2}{6}\sum_{r=1}^{\infty}\frac{c(n,r)}{r^2}, \tag{10}$$

$$\frac{\phi(n)}{n}=\frac{6}{\pi^2}\sum_{r=1}^{\infty}\left(\frac{\mu(r)}{J_2(r)}\right)c(n,r), \tag{11}$$

$$\frac{\Phi(n)}{n}=\frac{\pi^2}{15}\sum_{r=1}^{\infty}\left(\frac{\lambda(r)}{r^2}\right)c(n,r), \tag{12}$$

$$\frac{\psi(n)}{n}=\frac{15}{\pi^2}\sum_{r=1}^{\infty}\left(\frac{\mu^2(r)J_2(r)}{J_4(r)}\right)c(n,r). \tag{13}$$

The relations (10) and (11) are classical results of Ramanujan.

REMARK. In [2] the notion of almost periodicity is replaced by that of "almost parity," and the Wintner theory of arithmetical functions is reformulated and generalized in a manner that does not require additivity and periodicity properties.

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