DIFFERENTIABLE IMBEDDINGS

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1. **Terminology.** V^n and M^m will be differentiable manifolds of dimension n and m respectively; differentiable meaning always of class C^∞. For simplicity, we assume V compact and without boundary.

We shall have to consider several categories of maps:

(1) the category of continuous maps,

(2) the category of topological imbeddings,

(3) the category of topological immersions: a map $f\colon V \to M$ is a topological immersion of V in M if the restriction of f to some neighborhood of each point of V is an imbedding,

(4) the category of differentiable immersions: a map $f\colon V \to M$ belongs to this category if f is differentiable of rank $n = \dim V$ everywhere,

(5) the category of differentiable imbeddings: a differentiable imbedding $f\colon V \to M$ is a topological imbedding which is also a differentiable immersion.

Two maps $f_0, f_1\colon V \to M$ in one of the preceding categories are said to be *homotopic* in this category, if there exists a map $F\colon V \times R \to M$ (called a *homotopy* from f_0 to f_1) such that $F| V \times \{0\} = f_0$, $F| V \times \{1\} = f_1$ and the associated map $(x, t) \to (F(x, t), t)$ of $V \times R$ in $M \times R$ belongs to the given category.

A homotopy in the category of differentiable imbeddings is also called a *differentiable isotopy* (cf. [4]).

2. **Existence theorem.** Many results have been obtained recently in the combinatorial case (cf. [2; 3; 10; 12; 13]).

The following theorem is in some sense a generalization of Whitney's theorems (cf. [6; 8]) and Wu's theorem (cf. [11]) and the differentiable analogues of the above results.

A space X is q-connected (q an integer) if its homotopy groups vanish in dimension less or equal to q (for $q < 0$, the condition is empty; for $q = 0$, X is connected; for $q = 1$, X is connected and simply connected, and so on).

THEOREM 1. *Let V^n and M^m be two differentiable manifolds which are respectively $(k-1)$-connected and k-connected. Then*

(a) *Any continuous map of V in M is homotopic to a differentiable*

imbedding if $m \geq 2n - k + 1$ and $2k < n$ (and to a differentiable immersion if $m \geq 2n - k$ and $2k < n$, V and M k-connected).

(b) *Two differentiable imbeddings of V in M which are homotopic as continuous maps are differentiably isotopic if $m \geq 2n - k + 2$ and $2k < n + 1$.*

COROLLARY. *Any two differentiable imbeddings of a differentiable homotopy n-sphere in R^m are differentiably isotopic if $m > 3(n+1)/2$.*

Any differentiable imbedding of the standard sphere S^n in R^m can be extended to a differentiable imbedding of the unit ball B^{n+1} if $m > 3(n+1)/2$.

3. Sketch of the proof. First approximate the continuous map of V in M by a "generic" differentiable map f (cf. [5]). That means, if $2m > 3n$, that (a) f has no triple points, (b) at a double point $y = f(x_1) = f(x_2)$, $x_1 \neq x_2$, the images by df of the tangent spaces to V at x_1 and x_2 span the tangent space to M at y, (c) at a singular point $x \in V$ (i.e., where rank $f < n$), there exist local coordinates (x_i) around x and (y_j) around $y = f(x)$ such that f is given up to the second order by the equations:

$$y_i = x_i, \ 1 \leq i \leq n, \ y_n = \overset{2}{x_n}, \ y_{n+j} = x_n x_j, \ 1 \leq j \leq m - n.$$

Under these conditions, the double points of f form in M a $(2n - m)$-submanifold D; its boundary S is the image by f of the singular points.

Construct a differentiable function ϕ on D, $0 \leq \phi \leq 1$, which is generic (i.e., with nondegenerate singular points), equal to zero on S.

We want now to construct step by step a continuous deformation f_t of f which pushes away the double points along D: the double points of f_t will be contained in the submanifold of D consisting of the points y where $\phi(y) \geq t$. The essential difficulty occurs when one has to cross a singular point y of ϕ of index q. To describe the deformation at y, one constructs a model which is an analogous map f_0 of R^n in R^m, and a deformation of f_0 which is constant outside small neighborhoods of a $(q+1)$-cell in R^n and a $(q+2)$-cell in R^m. Then one has to prove the existence of diffeomorphisms h and h' of these neighborhoods into some neighborhoods of $f^{-1}(y)$ and y in V and M respectively such that $fh = h'f_0$. This is the main technical difficulty; we have to suppose here that $\pi_q(V) = 0$, $\pi_{q+1}(M) = 0$ and $2q + 2 < n$ and we use essentially results of Whitney on stability of singular points in this range (cf. [7]).

The method is analogous in the case of isotopy. We use, moreover, the covering homotopy property for imbedding spaces (see Thom [4, pp. 3–4]).

4. **Approximation theorem.** The preceding method leads directly to the following

THEOREM 2. (a) *Any topological imbedding (respectively topological immersion) of V^n in M^m can be approximated by a differentiable imbedding if $m \geq 3(n+1)/2$ (resp. a differentiable immersion if $m > 3n/2$).*

(b) *Let f_0, f_1 be two differentiable imbeddings (resp. immersions) of V in M. Any homotopy in the category of topological imbeddings (resp. immersions) can be approximated by a differentiable isotopy if $m > 3(n+1)/2$ (resp. a homotopy in the category of differentiable immersions if $m > (3n+1)/2$).*

This means that, in some "stable range," the classification of differentiable imbeddings (or immersions) of V in M does not depend on the differentiable structures of V and M.

5. **Obstruction theory.** The preceding technique uses no fact in algebraic topology except that the lower dimensional homotopy groups of the Stiefel manifolds are trivial. In other words we have considered only cases where the obstructions trivially vanish.

In the case of an imbedding of a complex in R^m, A. Shapiro has built up an obstruction theory in the stable range (mainly unpublished, see [3] for the first obstruction and [10]); on the other hand, W. T. Wu has initiated the study of isotopy (see [12]).

Any imbedding f of a space V in R^m gives a continuous map f^2 of the space $V \times V - V$ (where V is identified with the diagonal of $V \times V$) into the unit sphere $S^{m-1} \subset R^m$: for two distinct points x_1, x_2 of V, $f^2(x_1, x_2)$ is the unit vector $(f(x_2) - f(x_1))/|f(x_2) - f(x_1)|$. It is clear that f^2 is equivariant with respect to the symmetry which exchanges the factor of $V \times V - V$ and the antipodal map of S^{m-1}. If f_0 and f_1 are two homotopic imbeddings, then f_0^2 and f_1^2 are equivariantly homotopic.

It is well known that the equivariant maps of $V \times V - V$ in S^{m-1} are in 1-1 correspondence with the sections of the following sphere bundle E. Let V^* be the reduced symmetric square of V (i.e., the space obtained from $V \times V - V$ by identification of (x_1, x_2) and (x_2, x_1)); the orbit space of the cyclic group of order 2 acting on $(V \times V - V) \times S^{m-1}$ by symmetry on both factors is a bundle E with base V^* and fiber S^{m-1}.

If one combines the Shapiro-Wu theory valid in the combinatorial case with the preceding technique, one obtains the following formulation:

THEOREM 3. *The differentiable isotopy classes of differentiable imbeddings of a compact manifold V^n in R^m are in 1-1 correspondence*

with the homotopy classes of continuous sections of the sphere bundle E over V^*, provided $m > 3(n+1)/2$.

A similar statement for differentiable immersions of V^n into M^m can be proved by the same methods. A different proof using the Smale-Hirsch theory of immersions (cf. [1]) will appear in a joint paper of M. Hirsch and the author.

Theorems 2 and 3 (including immersions) should also be true in the category of imbeddings of complexes in manifolds.

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