ON A PROBLEM OF KLEENE'S

BY R. O. GANDY

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THEOREM I.[1] *The class of functions of hyperdegree strictly less than* $\mathbf{0}'$ *provides a basis for the predicate* $(E\alpha)(x)\overline{T}_1^1(\bar{\alpha}(x), a, a)$, *and hence for all predicates which belong to* Σ_1^1.

This theorem settles a problem left open by Kleene in [4]. To prove it we observe that Theorem XXVI of [3] relativises uniformly to an arbitrary function α (see Theorem XXVII of [3]). Thus there is a recursive $K(u, v)$ such that:

(i) $(\alpha)(E\beta)(x)K(\bar{\alpha}(x), \bar{\beta}(x))$;
(ii) $(\alpha)(\beta)_{\beta\in HA(\alpha)}(\bar{x})K(\bar{\alpha}(x), \bar{\beta}(x))$,

where $HA(\alpha)$ denotes the class of functions hyperarithmetic in α.

Suppose a satisfies the predicate $(E\alpha)(x)\overline{T}_1^1(\bar{\alpha}(x), a, a)$; then, by (i), there exist functions α, β such that

$$\text{(A)} \qquad (x)\overline{T}_1^1(\bar{\alpha}(x), a, a) \quad \& \quad (x)K(\bar{\alpha}(x), \bar{\beta}(x)).$$

And we can construct such functions recursively in O (cf. 5.5 (5) of [5]). But if $O\in HA(\alpha)$ then also $\beta\in HA(\alpha)$, which would contradict (ii). Hence there is an α of hyperdegree strictly less than $\mathbf{0}'$ such that $(x)\overline{T}_1^1(\bar{\alpha}(x), a, a)$; and this proves the theorem.[2]

By an obvious elaboration of the above argument we can construct, recursively in O, an infinite sequence of non-hyperarithmetic functions α_i such that $\boldsymbol{\alpha}_1 \not< \boldsymbol{\alpha}_0$, $\boldsymbol{\alpha}_2 \not< \boldsymbol{\alpha}_0 \cup \boldsymbol{\alpha}_1$, $\cdots$ (where bold face type denotes a hyperdegree). Thus we can prove

COROLLARY 1. *There are infinitely many distinct hyperdegrees lying between* $\mathbf{0}$ *and* $\mathbf{0}'$.

COROLLARY 2.[3] *If a* Π_1^1 *set of axioms for second-order arithmetic has an* ω*-model, then it has an* ω*-model whose functions are all of hyperdegree strictly less than* $\mathbf{0}'$.

[1] For notations used see [2; 3; 6]; in particular we use boldface type for hyperdegrees. $\mathbf{0}'$ is the hyperdegree of O.

[2] G. Kreisel points out that a similar construction may be used to prove a result of J. R. Shoenfield's (*Degrees of models*, Amer. Math. Soc. Notices vol. 6 (1959) p. 530): the functions whose degree is strictly less than the degree $\mathbf{0}'$ provide a basis for Σ_1 predicates in which the existential function quanfiier is bounded by a given recursive function. I am also indebted to Kreisel for suggesting Corollary 2 below.

[3] By a "Π_1^1 set of axioms" we mean a set of formulae whose Gödel numbers form a Π_1^1 set. An "ω-model" is a model which is standard with respect to the natural numbers.

Without loss of generality we may suppose that the only function variables of second-order arithmetic are variables for functions of a single argument. Let β be a function of two arguments; then the condition that the set of functions $\lambda x \cdot \beta(x, i)$ $(i=0, 1, \cdots)$ provides a denumerable ω-model for a Π_1^1 system of axioms can be expressed in the form $(E\alpha)(x)R(\alpha, \beta, x)$, with recursive R. The corollary now follows immediately from the theorem.

This corollary shows that not all sets which are representable (as defined in [2]) occur in every ω-model. It also shows that minimum ω-models[4] (if such there be) for inductively defined sets of axioms will not contain functions of hyperdegree $\mathbf{0}'$, and so cannot be used (in the way anticipated by Wang in [7]) to extend the concept of predicative set to include, say, O.

It would be of considerable interest if one could strengthen Theorem I by proving the existence of minimal bases closed with respect to hyperarithmetic operations. The basis $\mathcal{B}$ given in Theorem I is certainly not minimal. Indeed, given any non-hyperarithmetic function α, Theorem I of [1] shows that one can omit from $\mathcal{B}$ all functions in which α is recursive. And by a refinement of the construction used in [1] it can be shown that all functions whose hyperdegree is comparable with $\boldsymbol{\alpha}$ may be omitted from $\mathcal{B}$.

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THE UNIVERSITY, LEEDS, ENGLAND

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[4] M is a minimum ω-model for the set of axioms A if it is an ω-model for A, and no function of M provides an ω-model for A.