MARTINGALES OF BANACH-VALUED RANDOM VARIABLES[1]

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1. Let $(\Omega, \mathcal{B}, P)$ be a probability space and let $\mathfrak{X}$ be a Banach space. Let $X(\omega)$ be a Bochner-integrable function on $(\Omega, \mathcal{B}, P)$ taking values in $\mathfrak{X}$. Let $\mathfrak{F}$ be a Borel-field contained in $\mathcal{B}$. We define $E(X|\mathfrak{F})$, "a conditional expectation of $X(\omega)$ relative to $\mathfrak{F}$" as a Bochner-integrable function on $(\Omega, \mathcal{B}, P)$ (henceforth called random variable or r.v.) such that $E\{X|\mathfrak{F}\}$ is measurable $\mathfrak{F}$ and that

$$\int_A E(X \mid \mathfrak{F})dP = \int_A XdP, \qquad A \in \mathfrak{F},$$

where the integrals are Bochner-integrals.

Let I be a subset of the set of all integers and let $X_i(\omega)$, $i \in I$ be Bochner-integrable $\mathfrak{X}$-valued r.v.'s. Let $\mathfrak{F}_i \subset \mathcal{B}$, $i \in I$ be a set of Borel-fields such that

$$\mathfrak{F}_i \subset \mathfrak{F}_j \qquad \text{if } i < j.$$

The triple $\{X_i, \mathfrak{F}_i, i \in I\}$ will be called a martingale if whenever

$$i < j,\ i, j \in I,$$
$$X_i = E(X_j \mid \mathfrak{F}_i), \text{ a.e.}$$

If $Z(\omega)$ is any Bochner-integrable r.v. and if we define

$$X_i = E(Z \mid \mathfrak{F}_i)$$

then it follows directly from above that $\{X_i, \mathfrak{F}_i, i \in I\}$ is a martingale.

By $L_p(\Omega, \mathfrak{F}, P, \mathfrak{X})$, $1 \leqq p \leqq \infty$, we shall denote the class of those $\mathfrak{X}$-valued functions $X(\omega)$ which are strongly-measurable w.r.t. $\mathfrak{F}$ and such that $\|X(\omega)\|$ is in L_p. If we define the "norm" of $X(\omega) \in L_p(\Omega, \mathfrak{F}, P, \mathfrak{X})$, denoted by $[X]_p$, to be the L_p norm of $\|X(\omega)\|$, then, as is well-known, $L_p(\Omega, \mathfrak{F}, P, \mathfrak{X})$ becomes a Banach space.

A set of complex-valued r.v.'s $X_t(\omega)$, $t \in T$ will be said to be "restrictedly uniformly integrable"[2] if for any $\epsilon > 0$ there exists a $\delta > 0$ such that

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[2] I am indebted to Professor Doob for having suggested this terminology in one of his letters to me. I also thank him for pointing out to me that our Theorem 1 had been obtained by Mrs. Moy before, and that Mr. Scalora obtained results somewhat similar to ours in his doctoral dissertation.

$$\int_A |X_t(\omega)|\, dP < \epsilon,$$

wherever $P(A) < \delta$. Notice that for general probability measures (e.g. those measures which contain atoms) restrictedly uniform integrability is a weaker condition than ordinary uniform integrability.

2. We shall state our main results, deferring the proofs to a later publication.

THEOREM 1. *For any Bochner-integrable r.v. X and Borel-field $\mathfrak{F}$ $E(X|\mathfrak{F})$ exists and is unique except for sets of P-measure* 0.

In passing we state that the usual properties of a Bochner-integral are valid for $E(X|\mathfrak{F})$ e.g.

$$\|X\| \leqq E(\|X\| \mid \mathfrak{F}) \text{ a.e., etc.}$$

THEOREM 2. *Let $\{X_n, \mathfrak{F}_n, n \geqq 1\}$ be a $\mathfrak{X}$-valued martingale such that*

$$X_n = E(Z \mid \mathfrak{F}_n), \qquad n \geqq 1,$$

where

$$Z \in L_p(\Omega, \mathfrak{B}, P, \mathfrak{X}), \qquad 1 \leqq p < \infty, \mathfrak{X} \textit{ arbitrary.}$$

Then

$$\lim_{n\to\infty} [X_n - X_\infty]_p = 0 \tag{1}$$

where $X_\infty = E(Z|\mathfrak{F}_\infty)$ and

$$\mathfrak{F}_\infty = \textit{Borel-field generated by } \bigcup_{n=1}^{\infty} \mathfrak{F}_n.$$

THEOREM 3. *Let $\{X_n, \mathfrak{F}_n, n \leqq -1\}$ be a $\mathfrak{X}$-valued martingale, $\mathfrak{X}$-arbitrary, and let $X_{-1} \in L_p(\Omega, \mathfrak{B}, P, \mathfrak{X})$ $1 \leqq p \leqq \infty$. Then*

$$\lim_{n\to\infty} [X_{-\infty} - X_{-n}]_p = 0 \tag{2}$$

where

$$X_{-\infty} = E(X_{-1} \mid \mathfrak{F}_{-\infty})$$

and

$$\mathfrak{F}_{-\infty} = \bigcap_{n=1}^{\infty} \mathfrak{F}_{-n}.$$

THEOREM 4. *Let $\mathfrak{X}$ be a reflexive Banach space and let $\{X_n, \mathfrak{F}_n, n \geqq 1\}$ be a $\mathfrak{X}$-valued martingale such that*

$$X_n \in L_p(\Omega, \mathcal{B}, P, \mathfrak{X}) \qquad n \geqq 1, 1 < p < \infty$$

and

$$[X_n]_p < C.$$

Then there exists $X_\infty \in L_p(\Omega, \mathcal{B}, P, \mathfrak{X})$ such that

$$\lim_{n\to\infty} [X_n - X_\infty]_p = 0 \tag{3}$$

and

$$X_n = E(X_\infty \mid \mathfrak{F}_n).$$

THEOREM 5. *Let $\{X_n, \mathfrak{F}_n, n \geqq 1\}$ be a $\mathfrak{X}$-valued martingale where $\mathfrak{X}$ is reflexive and let $\|X_n\|$, $n \geqq 1$ be restrictedly uniformly integrable. Then there is a Bochner-integrable r.v. X_∞ such that*

$$\lim_{n\to\infty} [X_n - X_\infty]_1 = 0. \tag{4}$$

THEOREM 6. *Theorem 2 remains valid if we replace* (1) *by*

$$\lim_{n\to\infty} X_n = X_\infty \text{ a.e.}$$

THEOREM 7. *Theorem 3 remains valid if we replace* (2) *by*

$$\lim_{n\to\infty} X_{-n} = X_{-\infty} \text{ a.e.}$$

THEOREM 8. *Theorems 4 and 5 remain valid if we replace* (3) *and* (4) *by*

$$\lim_{n\to\infty} X_n = X_\infty \text{ a.e.}$$

THEOREM 9. *Let $\{X_n, \mathfrak{F}_n, n \geqq 1\}$ be a $\mathfrak{X}$-valued martingale where $\mathfrak{X}$ is reflexive and let* >

$$E\left(\sup_{n\geqq 0} \|X_n - X_{n-1}\|\right) < +\infty, \qquad X_0 \equiv 0.$$

Then

$$\lim_{n\to\infty} X_n(\omega) \text{ exists whenever}$$

$$\omega \in \left\{\omega : \sup_n \|X_n(\omega)\| < +\infty\right\}.$$

THEOREM 10. *Let* $\{X_n, \mathfrak{F}_n, n \geqq 1\}$ *be a* $\mathfrak{X}$*-valued martingale where* $\mathfrak{X}$ *is reflexive. If* $E(\|X_n\|) < C$*, independent of* n*, then there exists a Bochner-integrable r.v.* $X_\infty(\omega)$ *such that* $X_n(\omega)$ *converges weakly to* $X_\infty(\omega)$ *a.e.*

THEOREM 11. *If* $\mathfrak{X}$ *is not reflexive then none of the Theorems* 4, 5, 8, 9, *and* 10 *are valid.*

For the last theorem we consider $(\Omega, \mathfrak{B}, P)$ on the open unit interval with Lebesgue measure on Borel subsets, and consider $\mathfrak{X}$ to be the Banach space of all Lebesgue integrable functions on (0, 1). Let $\epsilon_\lambda(t) \in \mathfrak{X}$, $0 < \lambda < 1$ be as follows:

$$\begin{aligned} \epsilon_\lambda(t) &= 1, \qquad 0 < t \leqq \lambda, \\ &= 0, \qquad \lambda < t < 1. \end{aligned}$$

Let $\mathfrak{F}_n$ be the Borel-field generated by intervals

$$\left(\frac{m}{2^n}, \frac{m+1}{2^n}\right),$$

$0 \leqq m \leqq 2^n - 1$, $n \geqq 1$. Define

$$\begin{aligned} X_n(\omega) &= 2^n\{\epsilon_{(m+1)/2^n} - \epsilon_{m/2^n}\}, \qquad \omega \in \left(\frac{m}{2^n}, \frac{m+1}{2^n}\right), \\ &= 0, \qquad \text{elsewhere.} \end{aligned}$$

Then

$$\{X_n, \mathfrak{F}_n, n \geqq 1\}$$

is a martingale and

$$\begin{aligned} &\|X_n(\omega)\| \equiv 1 \text{ a.e.} \\ &E(\|X_n(\omega)\|^p) = 1, \qquad n \geqq 1, \\ &E\left(\sup_{n \geqq 0} \|X_n(\omega) - X_{n-1}(\omega)\|\right) = 1, \qquad X_0 \equiv 0. \end{aligned}$$

But if $\omega \neq p/2^q$ then $X_n(\omega)$ does not go to any limit either weakly or strongly. Actually no subsequence $X_{n_k}(\omega)$ converges weakly or strongly if $\omega \neq p/2^q$. Hence $X_n(\omega)$ does not converge in $L_1(\Omega, \mathfrak{B}, P, \mathfrak{X})$-mean either.

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