LOOP SPACES OF H-SPACES[1]

BY WILLIAM BROWDER

Communicated by Hans Samelson, June 6, 1960

Let Y be an H-space (a space Y with a continuous product $Y\times Y\to Y$ which has a unit element), Y arcwise connected and simply connected, and let $X=\Omega Y$, the space of loops of Y based at the unit. We will prove

THEOREM 1. *If $H^*(X)$ (the singular cohomology ring over the integers) is a finitely generated module over the integers, then X is of the same singular homotopy type as $K(G, 1)$ where G is a free abelian group ($K(G, 1)=S^1\times \cdots \times S^1=$the n-torus, where rank of $G=n$).*

Thus the loop space of an H-space Y is infinite dimensional unless $Y=K(G, 2)$, G free abelian.

The proof depends on Theorem 2 below.

Let p be a prime. Then the cohomology of an H-space Y over Z_p is a Hopf algebra (see [2]). If ψ: $Y\times Y\to Y$ is the multiplication in Y, then ψ: $H^*(Y; Z_p)\to H^*(Y; Z_p)\otimes H^*(Y; Z_p)$ is the diagonal map of the Hopf algebra, the product being the cup product.

Let A be a Hopf algebra over Z_p, ψ: $A\to A\otimes A$ the diagonal map, θ: $A\otimes A\to A$ the product. An element $x\in A$ is called primitive if $\psi(x)=x\otimes 1+1\otimes x$. An element $y\in A$ is called decomposable if $y\in\theta(\bar{A}\otimes\bar{A})$ where $\bar{A}$ is the subspace of A consisting of positive dimensional elements. Let $P(A)$ denote the primitive elements of A, $D(A)$ the decomposable elements of A, $Q(A)=A/D(A)$. Let ξ: $A\to A$ be defined by $\xi(x)=x^p$. Then $\xi(A)$ is a Hopf subalgebra of A.

We quote a theorem of Milnor and Moore [9].

THEOREM (MILNOR AND MOORE). *Let A be an associative, commutative Hopf algebra over Z_p with $A_0=Z_p$. Then the sequence $0\to P(\xi A)\to^f P(A)\to^g Q(A)$ is exact.*

Thus if $x\in P(A)\cap D(A)$, then $x=u^p$ for some $u\in A$.

Let $\mathcal{P}^i$ denote the ith Steenrod operation

$$\mathcal{P}^i\colon H^m(X; Z_p)\to H^{m+2i(p-1)}(X; Z_p) \qquad (p \text{ an odd prime}),$$

Sq^i denote the ith Steenrod square

$$Sq^i\colon H^m(X; Z_2)\to H^{m+i}(X; Z_2).$$

[1] This note was written while the author was a National Science Foundation Postdoctoral Fellow.

THEOREM 2. *Let Y be an H-space with $H^*(Y; Z_p) = P(y_1, \cdots, y_n, \cdots)$ = the ring of polynomials over Z_p generated by $y_1, \cdots, y_n, \cdots$, with* dim y_i *even for all i. Let $x \in H^{2m}(Y; Z_p)$ be a primitive element. If $p \neq 2$ and $m = p^r + k$ with $0 < k < p$, and $r > 0$, then $\mathcal{P}^{p^r+i}(x) \neq 0$ is indecomposable, $0 \leqq i < k$. If $p \neq 2$ and $1 < m < p$ $\mathcal{P}^i(x) \neq 0$ is indecomposable, $0 < i < m$. If $p = 2$, and $m = 2^{r-1}+1$ with $r > 1$, then $Sq^{2^r}(x)$ is indecomposable.*

PROOF. Since $H^*(Y; Z_p)$ is a polynomial ring $\mathcal{P}^m(x) = x^p \neq 0$. Let $p \neq 2$, $m = p^r + k$, $k < p$, $r > 1$. Then by the Adem relations [7]

$$\mathcal{P}^{p^r+k} = k!(\mathcal{P}^1)^k \mathcal{P}^{p^r}$$

so that $\mathcal{P}^{p^r+i}(x) = i!(\mathcal{P}^1)^i \mathcal{P}^{p^r}(x) \neq 0$ for $0 \leqq i \leqq k < p$. Now if α is an element of the Steenrod algebra and Y is an H-space, then $\alpha(P(H^*(Y; Z_p))) \subseteq P(H^*(Y; Z_p))$ since α is additive, so that $\mathcal{P}^{p^r+i}(x)$ is primitive in $H^*(Y; Z_p)$. If $\mathcal{P}^{p^r+i}(x)$ is decomposable then by the theorem of Milnor and Moore $\mathcal{P}^{p^r+i}(x) = u^p$. But $\mathcal{P}^1(\mathcal{P}^{p^r+i}) = (i+1)\mathcal{P}^{p^r+i+1}$ so that if $i < k$ $\mathcal{P}^1(u^p) \neq 0$. But $\mathcal{P}^1$ is a derivation by the Product Formula so that $\mathcal{P}^1(u^p) = pu^{p-1}(\mathcal{P}^1 u) = 0$ mod p. Hence $\mathcal{P}^{p^r+i}(x)$ is indecomposable in $H^*(Y; Z_p)$.

If $1 < m < p$, $p \neq 2$, we get from the Adem relations $\mathcal{P}^m = m!\,(\mathcal{P}^1)^m$, and we proceed similarly.

If $p = 2$ and $m = 2^{r-1}+1$ with $r > 1$ we have that $x^2 = Sq^{2^r+2}(x)$ and from the Adem relations

$$Sq^{2^r+2} = Sq^2 Sq^{2^r} + Sq^{2^r+1} Sq^1.$$

Since $H^*(Y; Z_2)$ is a polynomial ring on even dimensional generators $Sq^1 H^*(Y; Z_2) = 0$, for Sq^1 changes dimension by 1. Hence $Sq^{2^r+2}(x) = Sq^2 Sq^{2^r}(x)$ so that $Sq^{2^r}(x) \neq 0$ and is a primitive element. If $Sq^{2^r}(x) = u^2$ then $Sq^2(u^2) = (Sq^2 u)u + (Sq^1 u)(Sq^1 u) + u(Sq^2 u) = 0$ mod 2, since $Sq^1 \equiv 0$ in $H^*(Y; Z_2)$ (Sq^2 is a derivation on $H^*(Y; Z_2)$). Hence $Sq^{2^r}(x)$ is indecomposable. Q.E.D.

One can apply Theorem 2 to compute many Steenrod operations in the stable classical groups, using only the cohomology structure mod p and the fact that the classifying space is an H-space. We will use Theorem 2 to prove Theorem 1.

PROOF OF THEOREM 1. Let $\overline{X}$ = the universal covering space of X. Then it follows from the results of [3] that $H^*(\overline{X})$ is finitely generated. Further, $\overline{X} = \Omega \overline{Y}$ where $\overline{Y}$ is the 2-connected fibre space over Y (see [10]). Further $\overline{Y}$ is the fibre of a multiplicative fibre map of Y into $K(\pi_2(Y), 2)$, and hence $\overline{Y}$ is an H-space. We will show that $\overline{X}$ is acyclic, (i.e., that $H^i(\overline{X}) = 0$ for $i > 0$) and therefore X is a $K(\pi, 1)$, finite dimensional with π abelian finitely generated. Then π must be free abelian and the result will be achieved.

Therefore we will assume that $\pi_1(X)=0$ and show that X is acyclic.

If X is not acyclic, then $H^*(X)/\text{Torsion}$ is nontrivial (see Part 1 of Theorem 3 of [4] or see [5]), and hence $H^*(X)/\text{Torsion} = \Lambda(x_1, \cdots, x_n)$, the exterior algebra on odd dimensional generators $x_1, \cdots, x_n$ (see [2]). Since $H^*(X)$ is finitely generated, only a finite number of primes occur as torsion numbers of $H^*(X)$. Hence for almost all primes, in particular for all sufficiently large primes p, $H^*(X; Z_p) = (H^*(X)/\text{Torsion}) \otimes Z_p$. Therefore we have $H^*(X; Z_p) = \Lambda(x_1, \cdots, x_n)$ (identifying x_i with its image in $(H^*(X)/\text{Torsion}) \otimes Z_p = H^*(X; Z_p)$) for all sufficiently large p.

By a theorem of Borel (Theorem 13.1 of [2]) we have that $H^*(Y; Z_p) = P(y_1, \cdots, y_n)$ if the prime p is not a torsion number of $H(X)$, with $\dim y_i = \dim x_i + 1$. Let the y's be ordered so that $2k = \dim y_1 \leqq \dim y_i \leqq \dim y_n = 2m$, $1 \leqq i \leqq n$, and $k > 1$ since $\dim x_i > 2$ for all i.

Choose p so large that $2k+2(p-1) > 2m$ and $p > k > 1$, or in other words choose $p > \max(m-k-1, k)$, and large enough that p does not occur as a torsion number of $H^*(X)$. Then y_1 is primitive since it is in the first nonvanishing cohomology group of Y and we may apply Theorem 2 to $y_1 \in H^*(Y; Z_p)$. Hence $\mathcal{P}^1(y_1) \neq 0$ and is an indecomposable element in $H^*(Y; Z_p)$. But $\dim \mathcal{P}^1(y_1) = 2k+2(p-1) > 2m$, and all elements of $H^q(Y; Z_p)$ are decomposable if $q > 2m$. This is a contradiction, so X is acyclic. Q.E.D.

One might conjecture that if X is a homotopy commutative H-space and $H^*(X)$ is finitely generated then X is of the same singular homotopy type as $K(G, 1)$ with G a free abelian group. Araki, James and Thomas have shown that the usual multiplication on a compact Lie group G is not homotopy commutative unless G is a torus [1], and James [8] has shown that the spheres S^3 and S^7 have no homotopy commutative multiplications. It will be shown elsewhere [6] that if X is a homotopy commutative H-space with $H^*(X)$ finitely generated, then $H^*(X)$ has no 2-torsion. Hence the Lie groups which have 2-torsion (such as $SO(n)$ and the exceptional groups) have no homotopy commutative multiplications on them. It will also be shown in [6] that if X is homotopy associative and homotopy commutative, and $H^*(X)$ is finitely generated, then $H^*(X)$ has no torsion, so that $H^*(X) = \Lambda(x_1, \cdots, x_n)$.

In conclusion, it is pleasant to acknowledge the value of some conversations with J. Stasheff.

BIBLIOGRAPHY

1. S. Araki, I. M. James, and Emery Thomas, *Homotopy-Abelian Lie groups*, Bull. Amer. Math. Soc. vol. 66 (1960) pp. 332–334.

2. A. Borel, *Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts*, Am. of Math. vol. 57 (1953) pp. 115–207.

3. W. Browder, *The cohomology of covering spaces of H-spaces*, Bull. Amer. Math. Soc. vol. 65 (1959) pp. 140–141.

4. ———, *Homology and homotopy of H-spaces*, Proc. Nat. Acad. Sci. U.S.A. vol. 46 (1960) pp. 543–545.

5. ———, *Torsion in H-spaces*, to appear.

6. ———, *Homotopy commutative H-spaces*, to appear.

7. H. Cartan, *Sur l'itérations des opérations de Steenrod*, Comment. Math. Helv. vol. 29 (1955) pp. 40–58.

8. I. M. James, *Multiplication on spheres* (I), Proc. Amer. Math. Soc. vol. 8 (1957) pp. 192–196.

9. J. W. Milnor and J. C. Moore, *On the structure of Hopf algebras*, to appear.

10. J.-P. Serre, *Groupes d'homotopie et classes de groupes abeliens*, Ann. of Math. vol. 58 (1953) pp. 258–294.

CORNELL UNIVERSITY AND
THE MATHEMATICAL INSTITUTE, OXFORD