CLIFFORD PARALLELS IN ELLIPTIC $(2n-1)$-SPACE AND ISOCLINIC n-PLANES IN EUCLIDEAN $2n$-SPACE[1]

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An elliptic space is a projective space turned into a metric space by according a special role to an arbitrarily chosen but fixed non-degenerate imaginary hyperquadric. Let p_1, p_2 be any two points in the elliptic space. Then the distance between p_1, p_2 is defined as $(1/2(-1)^{1/2}) \log (p_1p_2\, q_1q_2)$, where q_1, q_2 are the two points at which the line $p_1 p_2$ intersects the hyperquadric and $(p_1p_2\, q_1q_2)$ denotes the cross-ratio of these four collinear points. It follows at once from the definition that (i) the distance (between any two real points) may be taken to be d or $\pi - d$ with $0 \leqq d \leqq \pi$, (ii) distances on the same straight line are additive, and (iii) the total length of any straight line is π.

It is well known that in an elliptic space of dimension 3, the concept of Clifford parallelism exists which has many interesting properties (see, for example, Klein [5]). A similar concept of parallelism for elliptic spaces of dimension $\geqq 3$ is the concept of Clifford-parallel $(n-1)$-planes in an elliptic space, El^{2n-1}, of dimension $2n-1$. We define this as follows:

In an El^{2n-1}, two $(n-1)$-planes A and B are said to be *Clifford-parallel* if the distance to B from any point in A is the same. The relation between two $(n-1)$-planes of being Clifford-parallel is reflexive, symmetric but not transitive. A set of $(n-1)$-planes in El^{2n-1} is called a *maximal set of mutually Clifford-parallel $(n-1)$-planes* if every $(n-1)$-plane in the set is Clifford-parallel to every other $(n-1)$-plane in the set, and if the set is not a subset of a larger set of mutually Clifford-parallel $(n-1)$-planes. A maximal set of mutually Clifford-parallel $(n-1)$-planes in El^{2n-1} is said to form a *foliation* (*partial foliation*) of El^{2n-1} if through each point of El^{2n-1} there passes one and only one (at most one) $(n-1)$-plane of the set.

Existence of maximal sets of mutually Clifford-parallel $(n-1)$-planes in any El^{2n-1} is established by the following theorem:

THEOREM 1. *In an* $\mathrm{El}^{2n-1}(n>1)$, *there are two or more maximal sets of mutually Clifford-parallel* $(n-1)$-*planes containing any given* $(n-1)$-*plane. If n is odd, there exist only* 1-*dimensional*[2] *maximal sets. If*

[1] Some of the results contained in this paper were obtained while the author was participating in a National Science Foundation Research Project at the University of Chicago in 1959.

[2] We call a set of $(n-1)$-planes p-dimensional if it depends on p parameters.

$n=2m$ (m = odd), there exist only 2-dimensional maximal sets. But if $n=2^s m$ (m = odd, $s>1$), then according as $s\equiv 1, 2, 3,$ or 0 (mod 4), *there exist and only exist maximal sets of dimension*

$$4, 8, 12, \cdots, 2s-6, 2s-2, 2s;$$

$$4, 8, 12, \cdots, 2s-4, 2s;$$

$$4, 8, 12, \cdots, 2s-2, 2s+2;$$

or

$$4, 8, 12, \cdots, 2s-4, 2s, 2s+1,$$

respectively.

Added in proof. The number of distinct (to within a motion or a motion followed by a reflection) p-dimensional maximal sets of mutually Clifford-parallel $(n-1)$-planes in an El^{2n-1} has been determined.

In an El^3, we have the classical results on Clifford-parallel lines. In an El^7, there are $4\infty^8$ maximal sets of mutually Clifford-parallel 3-planes containing any given 3-plane, and each of these maximal sets is of dimension 4 and forms a foliation of El^7. In an El^{15}, the maximal sets of mutually Clifford-parallel 7-planes are of dimensions 8 or 4, and each of the 8-dimensional maximal sets forms a foliation of El^{15}. In an elliptic space El^{2n-1} of any other dimensions (i.e. $2n-1\neq 3, 7, 15$), every maximal set of mutually Clifford-parallel $(n-1)$-planes forms only a partial foliation of the space El^{2n-1}.

The next three theorems show that in a certain sense a maximal set of mutually Clifford-parallel $(n-1)$-planes in El^{2n-1} is a linear set with the $(n-1)$-planes as elements.

THEOREM 2. *In an* El^{2n-1}, *let A, B be any two fixed Clifford-parallel $(n-1)$-planes, and C any $(n-1)$-plane Clifford-parallel to A and B such that distances between A, B, C are additive.*[3] *Then all such $(n-1)$-planes C form a 1-dimensional set of mutually Clifford-parallel $(n-1)$-planes and the distances between the $(n-1)$-planes of this set are additive.*

We call this 1-dimensional set, which obviously contains the two $(n-1)$-planes A and B, the *additive linear set* determined by the Clifford-parallel $(n-1)$-planes A, B. It plays a role similar to that of a straight line passing through two points.

[3] By this we mean that one of the three distances is equal to the sum of the other two.

THEOREM 3. *Let ξ be any maximal set of mutually Clifford-parallel $(n-1)$-planes in El^{2n-1}. If A, B are any two $(n-1)$-planes in ξ, then the additive linear set determined by A, B is contained in ξ.*

Bases of a special type exist in every maximal set of mutually Clifford-parallel $(n-1)$-planes in El^{2n-1}, as is seen in the following theorem:

THEOREM 4. *Let ξ be any p-dimensional maximal set of mutually Clifford-parallel $(n-1)$-planes in El^{2n-1}. Then there exist $p+1$, but not more than $p+1$, $(n-1)$-planes of ξ such that the distance between every two of them is $\pi/4$. Furthermore, if the distances from any $(n-1)$-plane of ξ to these $p+1$ $(n-1)$-planes are d_a $(0 \leqq a \leqq p)$, then $\sum_a \cos^2 2d_a = 1$. Conversely, for any given set of $p+1$ distances d_a such that $0 \leqq d_a \leqq \pi$ and $\sum_a \cos^2 2d_a = 1$, there exists a unique $(n-1)$-plane Clifford-parallel to each of these $p+1$ $(n-1)$-planes and at distances d_a from them, and this $(n-1)$-plane belongs to ξ.*

It is easy to see that the elliptic geometry of dimension $(2n-1)$ is equivalent to the geometry of m-planes $(1 \leqq m \leqq 2n-1)$ through a fixed point in a Euclidean $2n$-space E^{2n}. If we define two n-planes in E^{2n} to be *isoclinic with each other* when the angle between any line in one of the n-planes and its orthogonal projection in the other n-plane is always the same, then the Clifford parallelism in El^{2n-1} is equivalent to the concept of isoclinic n-planes in E^{2n}.

Isoclinic 2-planes in E^4, which do not necessarily pass through the same point, have been much studied, though seldom in conjunction with Clifford parallels in El^3 [6; 7; 9; 11]. An interesting connection with functions of one complex variable is the well-known theorem that a 2-dimensional surface of class C^2 in E^4 has the property that its tangent 2-planes are all mutually isoclinic iff the surface is an R-surface, i.e. a surface given in suitable rectangular coordinates (x, y, u, v) in E^4 by $u=u(x, y)$, $v=v(x, y)$, where $u(x, y)$ and $v(x, y)$ are the real and imaginary parts of an analytic function $f(x+iy)$. We try to find the higher dimensional analogues of such surfaces but obtain the following negative result:

THEOREM 5. *The only n-dimensional surfaces of class C^2 in E^{2n} $(n>2)$ whose tangent n-planes are all mutually isoclinic are the n-planes.*

To obtain the results stated above, we first determine all the maximal sets of mutually isoclinic n-planes in E^{2n}. Let $(x^1, \cdots, x^{2n})$ be rectangular coordinates in E^{2n}, and let an n-plane through the origin be given by the equation

$$x_1 = xA,$$

where $x = (x^1, \cdots, x^n)$ and $x_1 = (x^{n+1}, \cdots, x^{2n})$ are $1 \times n$ matrices, and A is an $n \times n$ matrix with constant real elements. If we denote by A also the n-plane whose equation is $x_1 = xA$, then a necessary and sufficient condition for the two n-planes A and B to be isoclinic with each other is that the matrix equation

$$(1 + AB')(1 + BB')^{-1}(1 + BA') = \rho^2(1 + AA')$$

be satisfied, where a dash indicates the transpose of a matrix and ρ is a suitable scalar which is equal to the cosine of the angle between the n-planes A and B. From this, we can prove that any maximal set of mutually isoclinic n-planes in E^{2n} containing the n-plane $x_1 = 0$ is congruent to a set of n-planes consisting of the n-plane orthogonal to $x_1 = 0$ and the n-planes whose equations are

$$x_1 = x(\lambda_0 + \lambda_1 B_1 + \cdots + \lambda_q B_q),$$

where the λ's are scalar parameters and the set $(B_1, \cdots, B_q)$ of real square matrices of order n is a maximal[4] real solution of the equations

$$(*) \qquad B_h + B_h' = 0, \qquad B_h^2 = -1, \qquad B_hB_k + B_kB_h = 0,$$
$$(h, k = 1, 2, \cdots; h \neq k).$$

The system (*) of equations has appeared in the literature in connection with the classical problem of A. Hurwitz's on composition of quadratic forms [**1**; **2**; **4**]. But for our purpose, a more detailed study of its real solutions than has hitherto been given is required. Using reductions by unitary similarity alone, we obtain all the maximal real solutions of (*), yielding as by-product a new and elementary proof of the Hurwitz-Radon theorem [**3**; **8**; **10**], which states that the equations (*), with $1 \leqq h, k \leqq p$, admit a solution in the field of complex numbers or the field of real numbers iff the pair of positive integers (n, p) has one of the following values:

$p = 2r + 1$ with $r \equiv 0$ or 3 (mod 4), and n is any multiple of 2^r;
$p = 2r + 1$ with $r \equiv 1$ or 2 (mod 4), and n is any multiple of 2^{r+1};
$p = 2r + 2$ with $r \equiv 3$ (mod 4), and n is any multiple of 2^r; and
$p = 2r + 2$ with $r \equiv 0, 1,$ or 2 (mod 4), and n is any multiple of 2^{r+1}.

[4] We say that $(B_1, \cdots, B_q)$ is *maximal* solution of (*) if it cannot be extended to a solution containing more matrices.

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