

RESEARCH ANNOUNCEMENTS

The purpose of this department is to provide early announcement of significant new results, with some indications of proof. Although ordinarily a research announcement should be a brief summary of a paper to be published in full elsewhere, papers giving complete proofs of results of exceptional interest are also solicited.

SOLUTION OF THE EQUATION $(pz+q)e^z = rz+s$

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An accurate knowledge of the roots of the equation in the title plays a part in the solution and application of linear difference-differential equations and in other theories. If $pr=0$, it is easy to transform our equation into $ze^z=a$, an equation dealt with in [1]. If $pr \neq 0$ and p, q, r, s are real, our equation transforms into one or other of the equations

$$(1) \quad (z+b)e^{z+a} = -(z-b)$$

and

$$(2) \quad (z+b)e^{z+a} = z-b,$$

where a, b are real.

We write $z=x+iy, w=u+iv$, where x, y, u, v are real. We cut the z -plane along the real axis from $z=-b$ to $z=b$ and write

$$(3) \quad w = z - \log \left\{ \frac{z-b}{z+b} \right\},$$

taking the logarithm real on the real axis outwith the cut, and writing $z=z(w)$ for the inverse function defined by (3). Clearly

$$\frac{dw}{dz} = \frac{z^2 - b^2 - 2b}{z^2 - b^2}$$

vanishes when $z=\zeta$, where $\zeta^2=b^2+2b$. We write

$$\omega = w(\zeta) = \zeta - \log \left\{ \frac{(\zeta-b)}{(\zeta+b)} \right\} = \zeta + \log(b+1+\zeta).$$

If $b > 0$, we have $\omega = \pm (b^2+2b)^{1/2} + \log \{ b+1+(b^2+2b)^{1/2} \}$. If $-2 < b < 0$ and we write $b+1 = \cos \theta, 0 < \theta < \pi$, we have $\zeta = \pm i \sin \theta, \omega = \pm i(\theta + \sin \theta)$, so that $0 < |\omega| < \pi$. If $b \leq -2$, we have $\omega = \pm \{ (b^2+2b)^{1/2} + \log (|b+1| - (b^2+2b)^{1/2}) \} \pm \pi i$. If $b = -2$, we have $\zeta = 0$ and $\omega = \pm i\pi$. We observe that, for any fixed b , there may be two or four values of ω , but all have the same modulus.

The inverse function $z(w)$ is uniquely defined if either (i) $|w| > |\omega|$

or (ii) $|v| > \pi$. Thus, if n is an integer such that $|n| \geq 2$ we may write

$$z_n = z(n\pi i - a).$$

We shall see that, for large n , $n\pi i - a$ is a good approximation to z_n . If $b > -2$ or if $b \leq -2$, $|a| > \Re(\omega)$, then $z(\pi i - a)$ has a real value $z^{(1)}$ and a nonreal value z_1 ; also $z(-\pi i - a)$ has a real value $z^{(1)}$ and a nonreal value z_{-1} . If $b \leq -2$ and $|a| \leq |\Re(\omega)|$, then $z(\pi i - a)$ has three values $z_{-1}, z_1, z^{(1)}$, all real, and $z(-\pi i - a)$ has the same three values. If $b > 0$, $|a| < \omega$, then $z(-a)$ has two (conjugate complex) values, z_0 and $z^{(0)}$. If $b > 0$, $|a| \geq |\omega|$ or if $b < 0$, then $z(-a)$ has two values, both real, z_0 and $z^{(0)}$, which we define so that that $|z_0| \geq |z^{(0)}|$.

The roots of (1) are

$$z^{(1)}; \dots, z_{-3}, z_{-1}, z_1, z_3, \dots$$

and those of (2) are

$$z^{(0)}; \dots, z_{-2}, z_0, z_2, z_4, \dots$$

If $b > 0$, $a = \omega$, the two roots $z_0, z^{(0)}$ coincide in a double root. If $b < -2$, $|a| = |\Re(\omega)|$, the two roots z_{-1}, z_1 coincide in a double root. If $b = -2$, $a = 0$, the three roots $z_1, z_{-1}, z^{(1)}$ coincide in a triple root at 0.

We define the polynomials $Q_m(b), R_s(a, b)$ by

$$Q_0(b) = 2b, \quad Q_m(b) = 2m(2m - 1) \int_0^b (b - \beta)(\beta + 2)\beta^{-1} Q_{m-1}(\beta) d\beta$$

for $m \geq 1$ and

$$R_s(a, b) = \sum_{0 \leq m \leq s/2} \binom{s}{2m} Q_m(b) a^{s-2m}.$$

In particular,

$$Q_1 = \frac{2}{3} b^3 + 4b^2, \quad Q_2 = \frac{2}{5} b^5 + \frac{16}{3} b^4 + 16b^3,$$

$$Q_3 = \frac{2}{7} b^7 + \frac{92}{15} b^6 + 40b^5 + 80b^4,$$

$$Q_4 = \frac{2}{9} b^9 + \frac{704}{105} b^8 + \frac{3136}{45} b^7 + \frac{896}{3} b^6 + 448b^5,$$

$$Q_5 = \frac{2}{11} b^{11} + \frac{2252}{315} b^{10} + \frac{6544}{63} b^9 + 704b^8 + 2240b^7 + 2688b^6.$$

If $|w| > |\omega|$, the inverse function $z(w)$ is uniquely defined and has the expansion

$$z = w - \sum_{m=0}^{\infty} Q_m(b)w^{-2m-1}.$$

Hence

$$(4) \quad z_n = n\pi i - a - \sum_{m=0}^{\infty} Q_m(b)(n\pi i - a)^{-2m-1},$$

provided that

$$(5) \quad n^2\pi^2 + a^2 > |\omega|^2.$$

For $b > 0$, $n = 0$, (5) is the condition that $z_0, z^{(0)}$ be real and different. For $n = 1$ and $b < 0$, (5) is the condition for z_1, z_{-1} to be nonreal; for $-2 < b < 0$ and $n \geq 1$, (5) is always satisfied.

If $z_n = x_n + iy_n$ and if

$$(6) \quad |n| \pi > \max |\omega \pm a|,$$

we can deduce from (4) that

$$(7) \quad x_n = -a + \sum_{k=0}^{\infty} \frac{(-1)^k R_{2k+1}(a, b)}{(n\pi)^{2k+2}}$$

and

$$(8) \quad y_n = n\pi + \sum_{k=0}^{\infty} \frac{(-1)^k R_{2k}(a, b)}{(n\pi)^{2k+1}}.$$

These expansions have the advantage over (4) that their terms are real.

It is not difficult to show that, if $z = x + iy$ is any nonreal root of (1) or (2), then

$$(9) \quad y^2 = -x^2 - b^2 - 2bx \coth(x + a).$$

We may thus use (7) to calculate x_n and determine y_n from (9).

Roughly speaking the successive terms of (7) and (8) diminish in size by a factor of about $\{\max |\omega \pm a| / n\pi\}^2$ and so very few terms are required if n is large. But the successive terms of (4) diminish by a factor of about $|\omega|^2 / (n^2\pi^2 + a^2)$ and so, if a is not small, (4) may be useful for values of n for which (7) and (8) are not. The disadvantage of calculating complex numbers is not here very great, since they are all powers of w^{-1} .

There will remain a known (usually small) number of roots which require special calculation. If we can find a rough approximation z' (say to the first decimal place) to one of these roots, we can improve

it as follows. If z', z'' are near one another and $w' = w(z')$, $w'' = w(z'')$, we have approximately

$$(10) \quad \begin{aligned} z'' &= z' + (w'' - w') [dz/dw]_{z=z'} \\ &= z' + (w'' - w') \left\{ 1 + \frac{2b}{z'^2 - \zeta^2} \right\}. \end{aligned}$$

Here w' may be calculated and $w'' = n\pi i - a$. Provided z' is not near a value of ζ , i.e. w'' is not near a value of ω , this method is practicable and may be repeated successively to obtain any required degree of approximation to z_n .

The first approximation to z_n may sometimes be found from the first term or two of (4), even when the series does not converge rapidly enough to provide a convenient means of obtaining the accurate value. Alternatively, there is a variety of graphical methods. The most attractive seems to be the following.

We write $z = bZ$, $Z = X + iY$ in (1) and (2) and they become

$$e^{bZ+a} = \pm (Z - 1)/(Z + 1)$$

and this leads to

$$(11) \quad bX + a = \log \left| (Z - 1)/(Z + 1) \right|$$

and

$$(12) \quad bY = \arg (Z - 1) - \arg (Z + 1) + k\pi.$$

We now use a sheet of tracing paper over a master-diagram consisting of the family of circles which pass through $(1, 0)$ and $(-1, 0)$ and the orthogonal family. The intersection of the circles of the first family with appropriate lines parallel to the Y -axis give points of the curve (12). Similarly circles of the other family cut lines parallel to the X -axis to give points of (11). The intersections of the two curves give the required values of Z and so of $z_n = bZ$. The first approximation obtained by this means is usually sufficiently good to enable us to use (10) to obtain any required degree of accuracy.

There remains the case in which w lies near ω so that (10) cannot be used conveniently to improve a first approximation to z . In this case we have two or three nearly equal roots. If $b = -2$ and a is small, we have $z_1, z_{-1}, z^{(1)}$ small and nearly equal. If we write $\eta^3 = 12a$, so that η has three possible values, we have the three values of z given by

$$z = \sum_{n=0}^{\infty} C_n \eta^{2n+1},$$

where

$$1 = C_0 = -20C_1 = 50400C_3, \quad 3 = 2800C_2$$

and, for $n \geq 4$,

$$20n(2n + 3)C_n$$

$$= (n - 1)(2n - 1)C_{n-1} - 10 \sum_{k=2}^{n-2} \{n(2n + 3) + 4k(n - k)\}C_kC_{n-k}.$$

The case of two nearly equal roots arises (i) when $b > 0$ and $-a$ is near a value of ω or when b is just less than 0 and a is small, when $z_0, z^{(0)}$ are nearly equal and (ii) when b is just greater than -2 and a is small or when $b < -2$ and $-a$ is near a value of $\Re(\omega)$, when z_1, z_{-1} are nearly equal. If we write

$$\rho = b/\zeta, \quad \eta^2 = 2(w - \omega)/\rho,$$

so that η has two possible values, we have the two values of z given by

$$z = \zeta + \sum_{m=0}^{\infty} \rho A_m \eta^{m+1},$$

where

$$A_0 = 1, \quad 12A_1 = 3 + \rho^2, \quad 288A_2 = 9 - 6\rho^2 + 5\rho^4, \\ 1080A_3 = -9\rho^4 + 5\rho^6.$$

More generally, if we put $v = \rho^{1/2}$, we have $A_m = v^m B_m(v)$, where

$$B_{m+2}(v) = B_{m+2}(1) + \frac{1}{16(m + 3)} \int_1^v \mu \frac{\partial}{\partial \mu} \left\{ \frac{(1 - \mu^4)^2}{\mu^3} \frac{\partial}{\partial \mu} (\mu B_m(\mu)) \right\} d\mu$$

and $B_m(1) = (-1)^m c_{m+1}$, where the c_m are the numbers of the sequence defined and partly tabulated on p. 92 of [1].

Elsewhere [2] we apply a different method to calculate the real roots of (1) and (2) and to determine the least upper bound of the real parts of the roots. The latter problem is of importance in applications to stability problems.

REFERENCES

1. E. M. Wright, *Solution of the equation $ze^z = a$* , Bull. Amer. Math. Soc. vol. 65 (1959) pp. 89-93.
2. ———, *The real roots of a transcendental equation*, (in preparation).

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