this interesting work has been made accessible to the English speaking community.

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An introduction to differential geometry. By T. J. Willmore. New York, Oxford University Press, 1959. 10+317 pp. $5.60.

Despite the renewed interest in differential geometry in the last decade, there has been but one text in English published since the war suitable for undergraduates. That one, by Struik, limits itself to topics in the classical theory of curves and surfaces. This is unfortunate, since a student familiar only with vector methods would find the geometric content of many modern research papers inaccessible merely on account of notation and terminology. Yet, current propaganda to the contrary, many geometers hestitate to do entirely without "line segments with arrows on the ends," especially in undergraduate lectures. For them, and for mathematicians in other branches who wish to find out a little of what's going on in the field, Willmore's new text is highly recommended.

No doubt the author will be accused of falling between the three fires of vector, tensor, and differential form notations, and it is clear that a book this size cannot do justice to all three. What is remarkable is how much is here, and how pertinent it is to modern trends in differential geometry.

The book is divided into two parts. The first is devoted to surface theory and uses vector notation almost exclusively. Chapter I covers some standard local theory of curves in three space, through the fundamental existence theorem. Topics not usually found in such a treatment are the discussions of differentiability hypotheses, existence of arc length, existence of the osculating plane, and an appendix giving an existence proof for solutions of systems of ordinary linear differential equations.

Chapter II, *Local intrinsic properties of a surface*, is an excellent introduction to Riemannian geometry. There is a good discussion of local isometric correspondence between surfaces as motivation for the study of intrinsic properties. Geodesics, geodesic curvature, the Gauss-Bonnet formula, conformal mapping, and geodesic mapping are treated, and an appendix proves the existence of normal coordinates (which, incidentally, are mentioned by name nowhere in the book).

Principal curvatures, developables, minimal surfaces (a minimal treatment), and ruled surfaces are covered in Chapter III, *Local non-intrinsic properties of a surface.* The fundamental existence theorem for surfaces is proved, modulo the corresponding theorem on existence of solutions for systems of first order partial differential equations. (The latter does *not* appear in an appendix.)

It is Chapter IV, *Differential geometry of surfaces in the large*, which is a delight to the modern differential geometer. Several sphere characterizations are given. Then the intrinsic metrization of a surface is constructed, and a proof of the Hopf-Rinow theorem outlined. Hilbert's theorem on the impossibility of embedding complete analytic surfaces of constant negative curvature in Euclidean three-space receives a proof which is much more detailed than that found in Blaschke's Vorlesungen. A section is devoted to conjugate points on geodesics and Bonnet's theorem. Before relating the Gauss-Bonnet formula to the Euler characteristic, a detailed definition of two-dimensional manifolds is given. The chapter closes with discussions (without proof) of various questions in continuing analytic line elements, embedding, and rigidity.

Part 2 begins with a modern treatment of tensor algebra, including a brief section on exterior algebra. The next chapter, *Tensor calculus*, contains rigorous definitions of differentiable manifold and tangent vector. Affine connections are given in tensor and in form notation, and covariant and absolute derivation introduced.

Chapter VII, *Riemannian geometry*, presents the essentials of the general theory: geodesic curvature, curvature tensor, Schur's theorem. Definitions are given for Einstein, symmetric, harmonic, and recurrent Riemannian structures. The last two types are, of course, especially popular in England, their native land, so Willmore understandably devotes parts of the succeeding sections to questions centering around recurrent tensor fields, as an application of the theory of parallel vectors. With the section on integrable distributions, theorem-proving comes to an end and the rest of the chapter is descriptive in nature. Of particular interest to non-specialists should be the section describing Cartan's approach to differential geometry. (Perhaps, in view of the rigorous treatment of the exterior calculus given in this very book, the warnings against Cartan's method are overdone.) The equations of structure are derived, and the chapter closes with statements of some results in global Riemannian geometry.

The final chapter rewrites parts of Chapters II and III in tensor and differential form notation.

There are very few misprints. The proof that (a) implies (b) in the Hopf-Rinow theorem is abbreviated almost to the point of error, especially in view of the care the author takes elsewhere to state explicitly what he does not prove. The index is an improvement over the usual Oxford Press product (although, for example, Schur's theorem cannot be found with its help).

This reviewer believes several objections to the omission of topics stem from more than personal preference. For instance, nowhere is a group mentioned: holonomy and motions are equally absent; symmetric spaces are left dangling with a tensor definition. Despite the jacket blurb, it is hard to see how this book would be useful to physics or engineering students. But advanced undergraduates and beyond should find it an excellent introduction. Indeed, it is the only book in English of its kind.

LEON W. GREEN

Aufbau der Geometrie aus dem Spiegelungsbegriff. By Friedrich Bachmann. Die Grundlehren der mathematischen Wissenschaften, vol. 96, Berlin-Göttingen-Heidelberg, Springer, 1959. 15+311 pp. DM 49.80.

This remarkable book is essentially an elaboration of an idea of G. Thomsen (*The treatment of elementary geometry by a group-calculus*, Math. Gaz. vol. 17 (1933) p. 232). The idea is to let the symbol for a point A or for a line a be used also for the corresponding involutory isometry, namely, A is the reflection in the point (or rotation through π about the point), and a is the reflection in the line. If A and B are two points, AB is the translation along their join through twice the distance between them. If a and b are two intersecting lines, ab is the rotation about their point of intersection through twice the angle between them. The point A and line a are incident if the corresponding transformations are commutative. The same property applied to two lines makes them perpendicular. In this case, if the lines are a and b, $ab(=ba)$ is their point of intersection! If $abc=cba$, the three lines a, b, c belong to a pencil, and then the line $d=abc$ is another member of the same pencil, namely the line for which the transformation ab (which is a rotation if a and b intersect) is equal to dc.

A good instance of the power of this method is the following generalization of the notion of isogonal conjugacy with respect to a triangle (p. 16): "If $ba'c=a''$, $cb'a=b''$, $ac'b=c''$, while $a'b'c'$ is involutory, then also $a''b''c''$ is involutory. *Proof.* $a''b''c''=ba'c\cdot cb'a\cdot ac'b=(a'b'c')^b$."

One soon begins to realize that such a geometry is not necessarily Euclidean. It is more like the "absolute" geometry of Bolyai, in which a pencil of lines having a common perpendicular is not necessarily the same as a pencil of parallels. In fact, the geometry may be regarded as a special kind of abstract group whose generators, called