BOOK REVIEWS

Ramification theoretic methods in algebraic geometry. By Shreeram Abhyankar, Princeton, Princeton University Press, 1959. 7+96 pp. \$2.75.

Abhyankar uses certain special definitions. A *local ring* (R, M) is any ring R with unit having a single maximal ideal M. Here R need not be Noetherian. A *semi-local ring* $(S: M_1, \cdots, M_t)$ is defined similarly. Finally, a domain A is called *normal* if it is integrally closed in its quotient field.

The basic situation of the book finds a normal local domain (R, M) with field of quotients K, and a finite algebraic extension K' of K. Then the integral closure S of R in K' is a semi-local domain $(S: N_1, \cdots, N_t)$ with a finite number of maximal ideals $N_1, \cdots, N_t$. The ideal MS is given by an expression of the form $MS = Q_1 \cap \cdots \cap Q_t = Q_1 \cdot \cdots \cdot Q_t$, where $Q_1, \cdots, Q_t$ are primary for $N_1, \cdots, N_t$ respectively. The normal local domains $R_1, \cdots, R_t = S_{N_1}, \cdots, S_{N_t}$ are said to *lie over* R. For each $i = 1, \cdots, t$, $R_i \cap K = R$. So R, which is uniquely determined by R_i, is said to *lie below* R_i. Let $M_i = N_i R_i$ be the maximal ideal of R_i. Then (R_i, M_i) is said to be *unramified* over (R, M) if the following two conditions are satisfied:

$$(1) \quad \begin{aligned} &\text{(a) } R_i/M_i \text{ is a separable extension of } R/M, \\ &\text{(b) } MR_i = M_i. \end{aligned}$$

Otherwise (R_i, M_i) is *ramified* over (R, M). The integral closure S is *unramified* over R if *all* the domains lying over R are unramified; otherwise it is ramified.

If S' is any domain with quotient field K' such that $R \subset S' \subset S$, then the *discriminant ideal* $D(S'/R)$ is defined to be the ideal of R generated by all the discriminants $D_{K'/K}(w_1, \cdots, w_n)$, where $w_1, \cdots, w_n$ is any basis of K'/K lying in S'. The domain S' is also semi-local with, say, maximal ideals $M_1', \cdots, M_s'$. We have then the general discriminant theorem of Krull which states that:

$$\sum_{i=1}^{S} [(S'/M_i'):(R/M)]_S \leqq [K':K]$$

with equality if and only if $D(S'/R) = R$. In the latter case, $S' = S$, S is a free R-module, and is unramified over R.

Suppose further that K'/K is a Galois extension (i.e., finite normal separable algebraic). Let $G = G(K'/K)$ be its Galois group. Then G permutes the domains $R_1, \cdots, R_t$ transitively. The splitting group

$G^s = G^s(R_j/R)$, the inertial group $G^i = G^i(R_j/R)$ and their corresponding fields K_j^s, K_j^i and domains R_j^s, R_j^i are defined in the customary fashion. These have the properties:

$$(2) \qquad R_j^s/M_j^s \simeq R/M, \qquad M_j^s = MR_j^s,$$

R_j is the only domain of K' lying over R_j^s.

R_j^i/M_j^i is isomorphic to the separable closure of R/M in R_j/M_j. It is normal over R/M with Galois group isomorphic to G^s/G^i, and

$$(3) \qquad M_j^i = M \cdot R_j^i.$$

However, in general these properties are not characteristic of G^s and G^i.

If R is a valuation ring, then the rings R_i are exactly the valuation rings of the extensions of this valuation to K'. However, a difficulty arises with respect to the definition of ramification given above. *For* (1) *is not what we normally mean when we say that the valuation ring* R_i *is unramified over* R. In that case we add the further condition that the natural map of the value groups $S_R \to S_{R_i}$ be onto. Only for real discrete valuations are those two concepts of ramification identical. Indeed, if for any valuation rings R, R' we were to have $M' \neq MR$, then the value group S_R would have a least positive element, which is certainly a very strong condition. Incidentally, this answers the author's question on page 59: we cannot conclude that $S_{R'} = S_R$ from $MR' = M'$.

The pièce de résistance of this book is Abhyankar's beautiful proof of the local uniformization theorem for dimension 2 and characteristic 0. The methods of ramification theory, together with Krull's theorem that in this case the inertial group of any valuation is abelian, reduce the problem to this: If K'/K is prime cyclic, ν' is the only extension of the valuation ν to K', ν has real value group and can be uniformized in K, then ν' can be uniformized in K'. This last statement, however, can easily be proved by brute force.

The book also contains a survey of valuation theory, especially for function fields, several theorems about the use of quadratic transformations on surfaces, and a rapid sketch of Zariski's proof of global two dimensional uniformization.

At several places in the main body of the book, the proofs of basic theorems are omitted and the reader is referred to the original sources. It is claimed that this is an aid to the student by the omission of tedious and complicated details. Nonsense! Any student will skip these details automatically on the first reading. The purpose for writing a textbook is exactly to have these details written down in one place rather than scattered over a dozen references. To omit them saves work for the author, not the student.

These lacunae, bad in any case, are doubly bad here. For most of the important omitted proofs could have been supplied in a few pages. The theorems of Krull on the structure of the splitting and inertial groups of valuations occupy less than ten pages in his original papers. As noted above, they are crucial for the proof of Abhyankar's theorem. Yet they are omitted, while more than ten pages are devoted to the theory of discriminants, which is not used at all.

Another serious omission is the Noether normalization theorem. Clearly one cannot discuss integral dependence of geometric rings without using this theorem. For this book, it would be best to prove it as a form of the

ZARISKI MAIN THEOREM. *Let (R, M) be a local ring such that R/M is infinite. Let (R', M') be a ring of quotients of a finitely generated extension $R[x_1, \cdots, x_t]$. Suppose that: $M' \supseteq M$, $[(R'/M'): (R/M)] < \infty$ and MR' is primary for M'. Then there exist $y_1, \cdots, y_t \in R'$ such that: R' is a ring of quotients of $R[y_1, \cdots, y_t]$, and $y_1, \cdots, y'$ are integral over R.*

The proof of this takes about a page. As immediate corollaries we get the local Noether normalization theorem, Propositions 3.12 and 3.18 (for R algebraic) and Lemma 3.18A of the book.

Another obvious lemma: If (R, M) is a Noetherian local domain, and S is finitely generated over R, then $[(S/MS): (R/M)] \geqq [S: R]$. As corollaries of this we get Theorem 1.44A and the following:

LEMMA. *If R is an algebraic normal domain with quotient field K, $[K': K] < \infty$ and $R' \subseteq K'$ lies over R, and if R'/R is unramified, then: K'/K is separable, and, for any Galois extension K^* of K containing K', and any R^* lying over R', K' is contained in the inertial subfield of R^*/R.*

Further, using the methods of Theorem 1.47, we can show that, for any ideal $I \subseteq R$ and any inertial domain R^i lying over R: $IR^i \cap R = I$. From this we see immediately that, in the situation of the lemma, R'/R is unramified if and only if K' is contained in the inertial subfield of R^*/R. This proves Lemma 1.41A and Proposition 3.18B (in the case of perfect ground fields).

The only other major omission is the unique factorization theorem for two dimensional regular local rings, for which a short elementary proof is known.

It is a shame that such a good book is marred by these omissions.

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