FOURIER-STIELTJES TRANSFORMS OF MEASURES ON INDEPENDENT SETS

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A subset E of the real line R will be called *independent* if the following is true: for every choice of distinct points $x_1, \cdots, x_k$ in E and of integers $n_1, \cdots, n_k$, not all 0, we have $n_1x_1 + \cdots + n_kx_k \neq 0$. The main result of this note is

THEOREM I. *There exists an independent, compact, perfect set Q in R which carries a positive measure σ whose Fourier-Stieltjes transform*

$$\int_{-\infty}^{\infty} e^{ixy} d\sigma(x) \qquad (y \in R)$$

tends to 0 *as* $|y| \to \infty$.

Sketch of proof. It is known ([5, Theorem IV] and [6, p. 25]) that there is a compact perfect set P in R which is not a basis (i.e., the set of all finite sums $\sum n_i x_i$, with $x_i \in P$ and integers n_i, does not cover R and hence has measure 0) but which carries a positive measure μ whose F.S. transform vanishes at infinity. A certain deformation of P will yield our set Q.

P is constructed as the intersection of a sequence of sets E_r which are unions of 2^r disjoint intervals $I_{j,r}$. Set $P_{j,r} = P \cap I_{j,r}$, for $1 \leqq j \leqq 2^r$.

REMARK 1. Since P is not a basis, the set of all points $w = (w_1, \cdots, w_k)$ in R^k such that $\sum_1^k n_j(x_j + w_j) = 0$ for some choice of $x_1, \cdots, x_k$ in P is, for each choice of integers $n_1, \cdots, n_k$, a closed set of measure 0 (a union of certain hyperplanes).

REMARK 2. Since there exists a function in $L^1(R)$ whose Fourier transform is 1 on $P_{j,r}$ and is 0 on the rest of P, we have

$$\lim_{|y| \to \infty} \int_{P_{j,r}} e^{ixy} d\mu(x) = 0 \qquad (1 \leqq j \leqq 2^r).$$

Choose a sequence $\{c_r\}$, $0 < c_r < 1$, such that $\prod_0^\infty c_r > 0$. Put $f_0(x) = x$, and inductively define a sequence of functions f_r on P, of the form

$$f_r(x) = x + w_{j,r} \qquad (x \in P_{j,r}). \tag{1}$$

Assume f_r is constructed, and has the property that the condition

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$$(\mathrm{A}_r) \qquad 0 < \sum_1^{2^r} |n_j|, \qquad |n_j| \leqq r, \qquad x_j \in P_{j,r}$$

implies

$$(\mathrm{B}_r) \qquad \sum_1^{2^r} n_j f_r(x_j) \neq 0.$$

By Remark 1 we can construct f_{r+1} so that (A_{r+1}) implies (B_{r+1}) and so that (A_r) implies

$$(2) \qquad \left| \sum_1^{2^r} n_j f_{r+1}(x_j) \right| > c_r \left| \sum_1^{2^r} n_j f_r(x_j) \right| .$$

Remark 2 implies that the functions

$$(3) \qquad g_r(y) = \int_P \exp\{ i f_r(x) y\} d\mu(x) \quad (r = 0, 1, 2, \cdots),$$

vanish at infinity, and it follows (again from Remark 1) that we can subject f_{r+1} to the further requirements that $|f_{r+1}(x) - f_r(x)| < 2^{-r}$ for $x \in P$ and that $|g_{r+1}(y) - g_r(y)| < 2^{-r}$ for all real y.

Define $f(x) = \lim_{r\to\infty} f_r(x)$. Our construction shows that no finite sum $\sum n_j f(x_j)$ can be 0 if the x_j are distinct points of P and the n_j are integers, not all 0. It follows that f is a homeomorphism of P onto an independent perfect set Q. Since the sequence $\{g_r\}$ converges uniformly, we have

$$(4) \qquad \lim_{|y|\to\infty} \int_P e^{if(x)y} d\mu(x) = 0.$$

The formula $\sigma(f(E)) = \mu(E)$ defines a measure σ on Q, such that

$$(5) \qquad \int_P e^{if(x)y} d\mu(x) = \int_Q e^{ity} d\mu(t),$$

and the theorem follows from (4).

We now list some consequences.

1. Let M be the Banach algebra of all bounded Borel measures on R, with convolution as multiplication, and let M_0 be the algebra of all $\mu \in M$ whose F.S. transforms vanish at infinity. It is known (see [4] for references) that M is not symmetric. Theorem I implies

THEOREM II. *M_0 is not symmetric.*[2]

[2] This answers a question raised by Irving Glicksberg.

This is proved from Theorem I by showing (either by Šreider's original method [6, pp. 21–22] or by a device due to J. H. Williamson [4, p. 234]) that there is a $\mu \in M_0$ such that the complex conjugate of its Gelfand transform (see [4]) is not the Gelfand transform of any member of M.

2. Call a compact set E in R a *Helson set* if every continuous function on E is the restriction to E of a F.S. transform. There exist perfect Helson sets [3] and every countable, independent, compact set is a Helson set. However, by [1] Theorem I implies

THEOREM III. *The independent perfect set Q is not a Helson set.*

It follows [3] that there is a bounded function whose spectrum lies in Q but which is not a F.S. transform; i.e., Q carries a "true pseudo-measure," in the terminology of [3].

3. Call a compact set E in R *strongly independent* if to every continuous function f on E, with $|f| \equiv 1$, and to every $\epsilon > 0$ there exists $y \in R$ such that $|f(x) - e^{iyx}| < \epsilon$ for all $x \in E$. This definition stems from Kronecker's theorem: every finite independent set is strongly independent.

Hewitt and Kakutani [2] have constructed strongly independent perfect sets. It is not hard to show that strongly independent sets are Helson sets, and we conclude:

THEOREM IV. *The independent perfect set Q is not strongly independent.*

4. Finally, we point out that Q furnishes an example of an independent perfect set which is a set of multiplicity (even in the restricted sense; see [7, pp. 344, 348]) for the convergence of trigonometric series, and that it is not a set of type N [7, p. 236], whereas every strongly independent set is of type N. In fact, to every strongly independent set E one can associate an increasing sequence of integers n_k such that $\sum \sin n_k x$ converges absolutely for all $x \in E$.

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ARITHMETIC PROPERTIES OF CERTAIN POLYNOMIAL SEQUENCES

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Consider the sequence of polynomials $\{u_n(x)\}$ that satisfy the recurrence

$$u_{n+1}(x) = (x + a(n))u_n(x) + b(n)u_{n-1}(x), \tag{1}$$

where $a(n)$, $b(n)$ are polynomials in n (and possibly some additional indeterminates) with integral coefficients. Moreover it is assumed that

$$u_0(x) = 1, \qquad u_1(x) = a(0), \qquad b(0) = 0. \tag{2}$$

The sequence $\{u_n(x)\}$ is uniquely determined by (1) and (2).

The writer [1, Theorem 1] has proved that if $m \geqq 1$, $r \geqq 1$, then $u_n(x)$ satisfies the congruence

$$\sum_{s=0}^{r} (-1)^s \binom{r}{s} u_{n+sm}(x)u_{(r-s)m}(x) \equiv 0 \pmod{m^{r_1}}, \tag{3}$$

for all $n \geqq 1$, where

$$r_1 = [(r + 1)/2], \tag{4}$$

the greatest integer $\leqq (r+1)/2$. In the present paper it is proved that $u_n(x)$ satisfies the simpler congruence

$$\sum_{s=0}^{r} (-1)^s \binom{r}{s} u_{n+sm}(x)u_m^{r-s}(x) \equiv 0 \pmod{m^{r_1}}, \tag{5}$$

where again r_1 is defined by (4). Also it is shown that (5) implies

$$\sum_{s=0}^{r} (-1)^s \binom{r}{s} u_{n+sm}(x)u_{k+(r-s)m}(x) \equiv 0 \pmod{m^{r_1}}, \tag{6}$$