ON GEODESICS THAT ARE ALSO ORBITS

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Let M be a (C^∞, connected) Riemannian manifold, i.e. for each $x \in M$, there is a positive-definite inner product on M_x, the tangent space to M at x. It is denoted by (,). Let X be a Killing vector field on M, i.e. X is the infinitesimal generator of a one-parameter group of isometries of M. A (C^∞) curve σ: $[0, a] \rightarrow M$ is an *orbit* of X if, for each $t \in [0, a]$, $X(\sigma(t))$, the value of X at $\sigma(t)$, is a multiple of $\sigma'(t)$, the tangent vector to σ at $\sigma(t)$.

Let $S(M)$ be the sphere bundle of unit tangent vectors to M. Define the real-valued function g on $S(M)$:

For $x \in M$, $v \in M_x$ with $(v, v) = 1$, $g(x, v) = (X(x), v)$.

THEOREM 1. *A $v_0 \in S(M)$, tangent to $x_0 \in M$, is a critical point for g (i.e. $dg = 0$ at v_0) if and only if the geodesic through x_0 tangent to v_0 is an orbit of X. In particular, if M is compact there is at least one geodesic of M which is also an orbit of X.*

Let R_{x_0} be the Riemann curvature tensor at x_0, i.e. for $v_1, v_2 \in M_{x_0}$, R_{x_0} is a linear transformation: $M_{x_0} \rightarrow M_{x_0}$ that is skew-symmetric with respect to the inner product. For $v \in M_{x_0}$, let $\Delta_v X$ be the covariant derivative of X in the direction of v. Define linear transformations A, B: $M_{x_0} \rightarrow M_{x_0}$ as follows:

$$A(v) = -R_{x_0}(v_0, v)(v_0), \qquad B(v) = \Delta_v X, \qquad \text{for } v \in M_{x_0}.$$

A is symmetric, B skew-symmetric with respect to the inner product. Let V be the vector space $v_0^\perp \oplus M_{x_0}$, with the direct-sum inner product. ($v_0^\perp$ is the orthogonal complement of v_0 in M_{x_0}).

Define linear transformations T, T_1: $V \rightarrow V$ as follows:

$$\begin{aligned} T(v \oplus v_1) &= v \oplus (A(v_1) + B(B(v_1))) \\ T_1(v \oplus v_1) &= B(v_1) \oplus -B(v). \end{aligned} \qquad \text{for } v \in v_0^\perp,\ v_1 \in M_{x_0}$$

Let Q and Q_1 be the quadratic forms on V

$$Q(v \oplus v_1) = (v \oplus v_1, T(v \oplus v_1)),$$
$$Q_1(v \oplus v_1) = (v \oplus v_1, T_1(v \oplus v_1)).$$

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THEOREM 2. *Suppose* $v_0 \in M_{x_0} \cap S(M)$ *is a critical point of* g. *Then,* $A(v_0) = B(v_0) = 0$. X *can be multiplied by a constant so that either*

$$\text{(a)} \qquad X(x_0) = -v_0 \qquad \text{or} \qquad \text{(b)} \qquad X(x_0) = 0.$$

The Hessian at v_0, *a quadratic form on the tangent space to* $S(M)$ *at* v_0, *has the same eigenvalues as* Q *in case* (a), *as* Q_1 *in case* (b).

In either case, there is at least one eigenvalue 0. This corresponds geometrically to the fact that every point of the curve in $S(M)$ through v_0 which is defined be the geodesic flow is a critical point for g. Morse theory, as extended by Bott to the critical-submanifold case [1], then applies to give relations between the topology of $S(M)$ and the number of orbit-geodesics.

Suppose M is compact; there is at least one critical v_0 for which the Hessian is positive semi-definite. One sees that case (b) does not hold unless X is identically zero. Then the form Q is positive semi-definite. The form $(v, B^2(v))$ is $\leqq 0$, since B is skew-symmetric, hence the form $(v, A(v))$ is always $\geqq 0$. But, this is the sectional curvature in the plane determined by v_0 & v. Thus, the mean value of the curvature in planes through v_0, i.e. the Ricci curvature of v_0, is $\geqq 0$. This gives a theorem of Bochner [3]:

If a compact Riemannian manifold has negative Ricci curvature, it has no Killing fields.

Precise computation of the eigenvalues of Q, hence of the index of g at the critical point v_0, depends on knowing $AB^2 - B^2A$. One can show that this depends linearly on the second covariant derivative of the curvature tensor.

It is plausible that these orbit-geodesics have a better chance of being closed than a geodesic chosen at random. For example, note that they cannot have self-intersections without being closed.

THEOREM 3. *The eigenvalues of the forms* Q & Q_1 *attached to an orbit-geodesic are independent of the point* x_0 *on the geodesic used to define it. If all eigenvalues but one are positive and* M *is compact, the orbit-geodesic is closed. For example, this is automatically true for a 2-dimensional compact manifold of positive curvature.* (*One can, as a matter of fact, prove that these must be surfaces of revolution, if they admit a Killing field.*)

Finally, we remark that the general reason that this machinery exists is that the space of geodesics of M, although not precisely a manifold in global structure, can be intuitively thought of as a symplectic manifold, i.e. a manifold of even dimension with a closed differential 2-form of maximal rank. A one-parameter group of isometries induces a one-parameter group of "symplectic automorphisms." But T. Frankel has shown the connection between fixed points of such groups and Morse theory [2]. A similar theory then holds for all Calculus of Variations problems, for it is well-known that the space of all "extremals" has such a "symplectic" structure. (This is essentially defined by the Integral Invariant of Poincaré.)

For example, the periodic solutions of the 3-body problem described by Lagrange in which the bodies are at the vertices of a rotating equilateral triangle are of the type we have been considering, i.e. are orbits of a one-parameter group of transformations leaving invariant the equations of motion.

Bibliography

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