NORMAL OPERATORS ON THE BANACH SPACE $L^p(-\infty, \infty)$. PART II: UNBOUNDED TRANSFORMATIONS[1]

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1. **Introduction.** Suppose $1<p<\infty$ throughout. The transformation D_p is defined by

$$D_p x = \text{derivative of } x$$

for all functions x in the set $\mathfrak{D}(D_p)$ of all locally absolutely continuous members of $L^p = L^p(-\infty, \infty)$ whose derivative belongs to L^p. The following result is prototypic: $(i/2\pi)D_p$ is a transformation (denoted P_p) that satisfies the spectral theorem in a sense to be described presently.

Let $\mathfrak{L}_p$ denote the norm-topology of L^p. Suppose that E is a *spectral resolution in* L^p (see §6), and let f be a function on $(-\infty, \infty)$. If T is a transformation of L^p with domain $\mathfrak{D}(T)$, then the relation

$$T \subseteq \int f(\lambda) \cdot E(d\lambda) \tag{1}$$

will mean that, for all functions x in $\mathfrak{D}(T)$, the Riemann-Stieltjes integral

$$\int_{-\infty}^{\infty} f(\lambda) \cdot E(d\lambda) x$$

converges to Tx in the topology $\mathfrak{L}_p$.

Let $\mathfrak{O}_p$ be the class of all linear transformations Q of L^p which give rise to a spectral resolution E^Q in L^p such that

$$Q \subseteq \int \lambda \cdot E^Q(d\lambda), \tag{i}$$

and

$$I_p \subseteq \int 1 \cdot E^Q(d\lambda), \tag{ii}$$

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where I_p is the identity-mapping on L^p. The basic result can now be more precisely formulated: $P_p \in \mathfrak{O}_p$. As pointed out by Dunford [2, p. 223], P_p is probably not a spectral operator (when $p \neq 2$). In case $p=2$, the relation $P_2 \in \mathfrak{O}_2$ comes from the spectral theorem ($\mathfrak{O}_2$ contains all self-adjoint linear transformations of L^2). Differential transformations of a more general sort also belong to $\mathfrak{O}_p$ (see 5.5).

2. **Applications.** Let A be a function of bounded variation on $R=(-\infty, \infty)$. The convolution transformation A_{*p} is defined for all x in L^p by the relation: $A_{*p}x = A * x$, where

$$(A * x)(\lambda) = \int_{-\infty}^{\infty} x(\beta) dA(\lambda - \beta) \qquad (\lambda \in R).$$

If the function g defined by

$$g(\lambda) = \int_{-\infty}^{\infty} e^{2\pi i\lambda\beta} dA(\beta) \qquad (\lambda \in R) \tag{2}$$

is of locally bounded variation, then

$$A_{*p} \subseteq \int g(\lambda) \cdot E^Q(d\lambda) \qquad (\text{where } Q = P_p). \tag{3}$$

Relation (3) is generalized by 5.3 below; it is closely connected with some results that Bade [1] has derived from different assumptions.[2]

Let H_p denote the Hilbert transformation (also called *conjugation operator*). Then $D_pH_p \in \mathfrak{O}_p$, and

$$D_pH_p \subseteq \int -(-\lambda)^{1/2} \cdot E^Q(d\lambda) \qquad (\text{where } Q = D_p^2); \tag{4}$$

here we assume $(\beta)^{1/2}=0$ whenever $\beta<0$. This provides an interpretation of the equality $D_pH_p = -(-D_p^2)^{1/2}$, which is stated to hold "in some sense" by Hille-Phillips (see the bottom of p. 579 in [5]).

Hille [4] has proved that D_pH_p is the infinitesimal generator of the Poisson semi-group $\{T(\alpha)_p : \alpha > 0\}$; this fact harmonizes with the relation

$$T(\alpha)_p \subseteq \int e^{\alpha\lambda} \cdot E^Q(d\lambda) \qquad (\text{where } Q = D_pH_p).$$

Theorem 5.4 (below) makes it easy to derive similar relations involving other semi-groups and their infinitesimal generators.[3]

[2] See also Exercise 4 in [2, p. 605].

[3] In this connection, I wish to correct an error in [7]: both occurrences of E_p^D in two first displayed formulas (on top of p. 271) should be replaced by E^Q, where $Q=P_p$.

3. **Unbounded multiplier-transformations.** As in [7], we denote by L^+ the intersection of the family $\{L^q: 1<q<\infty\}$, and again let $\mathfrak{E}$ be the class of all linear mappings K of L^+ into L^+ such that $\|K\|_q \neq \infty$ whenever $1<q<\infty$. The Fourier transformation and its inverse are written F_+ and F_-, respectively.

If f is a function on R, then $p(f: 1)$ is the class of all closed linear transformations T of L^p such that[4]

$$F_+(Tx) = (F_+x)\cdot f \qquad \text{(all } x \text{ in } \mathfrak{D}(T))$$

whereas $p(f: -1)$ is the class of all closed linear transformations T of L^p such that

$$F_-z \in \mathfrak{D}(T) \quad \text{and} \quad T(F_-z) = F_-(f\cdot z)$$

whenever z is a function in L^+ having compact support. Consider the class $t(f)_p$ of all endomorphisms of L^p whose restrictions to L^+ belong to $\mathfrak{E}\cap p(f: 1)$. The class $(t)_p$ described in [7] contains the class

$$(t')_p = \bigcup \{t(f)_p : f \text{ is of bounded variation on } R\}.$$

As in [7], we denote by $[\wedge(f)]_p$ the unique member of $t(f)_p$. If ϕ_α is the characteristic function of the interval $(-\alpha, \alpha)$, then $[\wedge(\phi_\alpha)]_p$ is is the Dirichlet transformation:

$$([\wedge(\phi_\alpha)]_p x)(\lambda) = \int_{-\infty}^{\infty} x(\beta)\frac{\sin 2\pi\alpha(\lambda-\beta)}{\pi(\lambda-\beta)}\, d\beta.$$

Let $\mathfrak{E}_p$ be the class of all linear transformations of L^p that commute with each member of the family $\{[\wedge(\phi_\alpha)]_p : \alpha>0\}$.

DEFINITION. *The symbol $p(f)$ will be used for the class of all T in $\mathfrak{E}_p$ such that, for all x in $\mathfrak{D}(T)$:*

$$[\wedge(\phi_\alpha)]_p Tx = [\wedge(f\cdot\phi_\alpha)]_p x \qquad (\textit{all } \alpha > 0).$$

4. The transformations considered in this paper belong to the class

$$(t'')_p = \bigcup \{p(f) : f \in [V]\},$$

where $[V]$ is the family of all complex-valued functions of locally bounded variation on $R=(-\infty, \infty)$. It is easily seen that $p(f)\supset t(f)_p$ when $f\in[V]$, so that $(t'')_p$ is an extension of $(t')_p$.

4.1. THEOREM. *Set $n_p=1$ when $p<2$, while setting $n_p=-1$ in case $p\geqq 2$. If $f\in[V]$, then*

$$p(f) \supset p(f: n_p) \cap \mathfrak{E}_p.$$

[4] In this article, "transformation of L^p" means "transformation whose domain and range are subsets of L^p."

4.2. THEOREM. *Suppose* $f\in[V]$ *and* $g\in[V]$. *If* $Q\in p(f)$ *and* $T\in p(g)$, *then* $QT\in p(f\cdot g)$.

5. **Main results.** Denote by $\mathcal{K}$ the set of all real-valued, piecewise monotone functions defined and continuous on R. Let $\mathcal{G}$ be the set of all functions g in $\mathcal{K}$ such that $|g(-\infty)| = \infty = |g(+\infty)|$. For example, any real polynomial belongs to $\mathcal{G}$.

5.1. THEOREM. *If* $T\in p(g)$ *and* $g\in\mathcal{G}$, *then* $T\in\mathfrak{O}_p$.

Set $I^1(\lambda)=\lambda$ for all $\lambda\in R$; in virtue of 4.1, the relation $P_p\in p(I^1)$ can be obtained by showing that $P_p\in p(I^1\colon n_p)\cap\mathfrak{C}_p$.

5.2. THEOREM. *If* $g\in[V]$ *and* $T\in p(g)$, *then*

$$T \subseteq \int g(\lambda)\cdot E^Q(d\lambda) \qquad (\textit{where } Q = P_p). \tag{5}$$

5.3. COROLLARY. *If* $g\in[V]$ *and* $T=[\wedge(g)]_p$, *then the relation* (5) *holds.*

This follows from 5.2 and $t(g)_p\subset p(g)$. The relation (3) comes from the fact that $A_{*p}=[\wedge(g)]_p$ when g is defined by (2).

5.4. THEOREM. *Suppose* $g\in\mathcal{G}$ *and* $Q\in p(g)$. *Let f be a continuous function on R such that the composition $f\circ g$ is a function ϕ in* $[V]$; *if* $T\in p(\phi)$, *then*

$$T \subseteq \int f(\lambda)\cdot E^Q(d\lambda).$$

Property (4) comes from 5.4. A repeated application of 4.2 yields the following

5.5. COROLLARY. *Let* $\{\alpha_0, \alpha_1, \alpha_2, \cdots, \alpha_n\}$ *be an arbitrary finite set of real numbers. Let T be the transformation defined by the relation*

$$Tx = \alpha_n(iD_p)^n x + \alpha_{n-1}(iD_p)^{n-1}x + \cdots + \alpha_1 iD_p x + \alpha_0 x$$

for all x in $\mathfrak{D}(D_p)$ whose successive derivatives D_px, $(D_p)^2x, \cdots, (D_p)^{n-1}x$ all belong to $\mathfrak{D}(D_p)$. Then $T\in\mathfrak{O}_p$, and the relation (5) *is satisfied for*

$$g(\lambda) = \alpha_n(2\pi\lambda)^n + \alpha_{n-1}(2\pi\lambda)^{n-1} + \cdots + \alpha_1(2\pi\lambda) + \alpha_0.$$

6. **The type of integration employed.** Let $\mathcal{B}$ be the Boolean set-algebra generated by the subintervals of $R=(-\infty, \infty)$. Suppose that E is a *spectral resolution in* L^p (that is to say, E is a homomorphism of the Boolean algebra $\mathcal{B}$ into the class of all the idempotent members

of $(t')_p$ that are self-adjoint in the sense of [7]). Let '$(\mathfrak{L}_p)$ lim' indicate convergence in the topology $\mathfrak{L}_p$. For example, (1) means that

$$Tx = (\mathfrak{L}_p) \lim_{\beta\to\infty;\alpha\to-\infty} \int_\alpha^\beta f(\lambda)\cdot E(d\lambda)x \tag{6}$$

for all x in $\mathfrak{D}(T)$. The meaning of the integral in (6) will now be indicated.

Let $\mathfrak{Z}$ be the set of all finite, disjoint families π of left-open, right-closed subintervals of R, such that π is a cover[5] of $[\alpha, \beta]$. Let $\mathfrak{S}$ be the set of all ordered pairs (π, σ) with $\pi \in \mathfrak{Z}$ and such that σ is a function on π with values σ_a in $[\alpha, \beta] \cap a^-$ (whenever $a \in \pi$).[6] We write

$$\Phi(\pi, \sigma) = \sum_{a\in\pi} f(\sigma_a)\cdot E(a)x.$$

The set $\mathfrak{S}$ is directed by the usual Riemann-Stieltjes ordering '$\gg$' (see [6, p. 79]), and we define

$$\int_\alpha^\beta f(\lambda)\cdot E(d\lambda)x = (\mathfrak{L}_p) \lim \{\Phi(\pi, \sigma) : (\pi, \sigma) \in \mathfrak{S}, \gg\}.$$

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[5] In the sense of [6, p. 49].

[6] Here a^- denotes the closure of a.

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