BOOK REVIEWS

Einführung in die Operative Logik und Mathematik by Paul Lorenzen, Berlin, Göttingen, Heidelberg, Springer-Verlag, 1955. 298 pp.

This book presents in more detail than earlier articles an important and original approach to the foundations of logic and of mathematics. Both disciplines are reinterpreted to have as subject matter operations according to schematic rules. Hence the name *operative*. For convenience, only rules for operating on symbol strings are considered, each list of such (primitive) rules determining a formal system (*Kalkül*). A (well-formed) rule, metarule, metametarule, . . . of a formal system K can be written in the form $A_1, \cdots, A_n \to A$, where enough dots are to be added above $\to$ to indicate the composition, where $\to$ is to be omitted when $n=0$, and where $A_1, \cdots, A_n, A$ are respectively formulas (i.e., strings consisting of atoms of K and of variables ranging over the atom strings of K), rules, metarules, . . . of K. It is *admissible in* K if respectively the derivable atom strings, admissible rules, admissible metarules, . . . of K are closed under it. *Consequential logic* is then a theory of *universal* admissibility, i.e., of forms of rules, metarules, . . . which are admissible in any system K. For one after another of the symbols $\wedge$, $\vee$, $\bigwedge_x$, $\bigvee_x$, $\neg$, $\equiv$, and ι_x admissibility in K (or a suitable extension of K) of rules, metarules, . . . containing these symbols is then defined, the technical device used depending on the desired formal properties of the symbol introduced. Intuitionistic and also classical systems of logic for the corresponding symbols are then likewise theories of universal admissibility, each theorem being the form of a universally admissible rule, metarule, ... For any K, x ranges here over the *objects* of K, given as the theorems of some K', $\equiv$ indicates similar shape of objects, and ι_x serves as description operator in forming *terms* with properties similar to objects.

Terms serving as abstractions are introduced in a similar manner. If $\rho(x, y)$ is an *abstract equality relation* in K in the sense that $\rho(x, x)$ and $\rho(x, z) \wedge \rho(y, z) \to \rho(x, y)$ are admissible in K, then admissibility in K is defined for those rules, metarules, . . . containing $\iota_y^\rho \rho(X, y)$ which satisfy a certain requirement of *compatibility* with ρ. $\iota_y^\rho \rho(X, y)$ might ordinarily be regarded as a name for the class of symbol strings equivalent under ρ to X, but here it is treated *as* a class, Cantor's notion of class being discarded as too vague. In particular, if "$=$" is some abstract equality relation in K, if $X(z)$ is an object form in the sense that substitution of objects for z yields objects, if $y(z)$ ranges over such object forms, and if $\rho(X, y)$ is the abstract equality relation $\bigwedge_z \cdot X(z)=y(z)$, then the term $\iota_y^\rho \rho(X, y)$ is called a *function*, and is often written as $\lambda_z X(z)$. Similarly, for systems K in which $\bigwedge_x \cdot A(x) \leftrightarrow B(x)$ or $\bigwedge_{x_1 \cdots x_n} \cdot A(x_1, \cdots, x_n) \leftrightarrow B(x_1, \cdots, x_n)$ respectively are abstract equality relations $\rho(A, B)$, the term $\iota_B^\rho \rho(A, B)$ is called a *class* or *relation* respectively, and often written as $\kappa_x A(x)$ or $\kappa_{x_1 \cdots x_n} A(x_1, \cdots, x_n)$ respectively.

Arithmetic is regarded as the body of rules, metarules, . . . admissible in a system such as E below, reflecting only the most elementary processes. First, the system Z whose only two (primitive) rules are $|$ and $k \rightarrow k|$ furnishes the *numbers*. Next, the system D whose only rules are $| = |$ and $k=l \rightarrow k| = l|$, where k and l range over numbers, yields theorems concerning similar shape of numbers. All of Peano's axioms then are true statements of admissibility in D. E may then be obtained, e.g., by adjoining to D two rules $\alpha(k, |, k|)$ and $\alpha(k, l, m) \rightarrow \alpha(k, l|, m|)$, so that $\iota_m \alpha(k, l, m)$ is a term, which can also be written as $k+l$, and by adjoining two similar rules for multiplication. The theories of rational and of algebraic numbers are obtained in a related manner.

The central topic of the book is the replacement of Cantor's notion of *set* by the notion of *formula of a* [denumerable] *language stratum* (*Schicht*) S_ϑ, the construction being such that formulas of, e.g., $S_{2\omega}$ can serve in place of Cantor's sets as an interpretation or *operative model* of analysis. If S_0 consists of the strings $X_1, X_2, \cdots$ formed from atoms $u_1, \cdots, u_i$ then the *formulas* of the *elementary language* S_1 *over* S_0 are obtained by adding to the formation rules of a first-order predicate calculus with individual terms $X_1, X_2, \cdots$ and relation signs $\rho_1, \cdots, \sigma_1, \cdots$ a rule for constructing further relation signs $|_\rho B$ which may enter into formulas, where B is an *induction scheme* for ρ. An induction scheme B for ρ is a conjunction of formulas $A_k \rightarrow \rho(X_{k1}, \cdots, X_{kn})$ in which the occurrences of ρ in A_k, if any, are such that, roughly speaking, if admissibility in a "system" K is a *definite* notion for the formulas not containing ρ, then for all $\rho(X_1, \cdots, X_n)$ admissibility in the "system" K' obtained by adding B to K is also definite, so that admissibility of $X_1, \cdots, X_n \in |_\rho B$ in K can then be equated with admissibility of $\rho(X_1, \cdots, X_n)$ in K'. Here K and K' may be *improper* systems since, e.g., those X_j such that $\bigwedge_y \sigma_k(y, X_j)$ is admissible in K may not form a recursively enumerable set so that the rule $\bigwedge_y \sigma_k(y, x) \rightarrow \rho(x)$ may be nonconstructive. The formation of $|_\rho B$ and, at least vaguely, the idea of replacing sets by formulas was suggested by Weyl's *Das Kontinuum*. New is the iteration of this method, $S_{\vartheta+1}$ being formed over S_ϑ as S_1 over S_0, its strings including those of S_ϑ and the formulas of $S_{\vartheta+1}$. When Θ is a limit ordinal, then S_Θ is the union of all S_ϑ, $\vartheta < \Theta$, and not the language over a class of atoms. The class of induction schemes in S_ϑ and that of formulas in S_ϑ can be represented in $S_{\vartheta+1}$, in the sense that, e.g., a relation sign φ can be introduced by induction schemes in $S_{\vartheta+1}$ such that $\varphi(X)$ is admissible in the "system" obtained from the induction schemes if and only if X is a formula of S_ϑ. Among the consequences are the following: There are classes of strings in S_0 which can be represented in $S_{\vartheta+1}$ but not in S_ϑ, so that power set becomes a relative notion. The classes representable in S_ϑ of strings in S_0 can be enumerated in $S_{\vartheta+1}$ but not in S_ϑ, so that enumerability becomes a relative notion. The axiom of choice holds in the sense that for any family M of nonempty sets which is representable in S_ϑ, a function can be represented in S_ϑ assigning to each set in M one of its elements (namely that with the least Gödel number).

Ordinary analysis is largely preserved. While orginally its "well-roundedness" was obtained by enlarging what "exists," it now is due to suitable restrictions of expressive means. The real numbers, obtained by abstraction from certain formulas of S_{Θ_1}, satisfy the Dedekind completeness theorem, while the theory of functions of real numbers in S_{Θ_1} requires an S_{Θ_2}, where Θ_1 and Θ_2 are limit ordinals and $\Theta_1 < \Theta_2$. Even though in an absolute sense each set is denumerable, a nontrivial Lebesgue measure can be defined, the role of denumerable sets now being played by those in S_{Θ_1}.—In the concluding part a sketch is given of how set- or model-theoretic considerations or ordinary mathematics concerning axiom systems are to be treated.

The reviewer believes that the book does not contribute directly to the formalist program nor to the intuitionist program but rather presents a legitimate and probably fruitful third approach to foundations. Its evident advantage over intuitionism is the preservation of classical logic and arithmetic and of much larger portions of the rest of mathematics, and over formalism the interpretation of these. Its evident disadvantage is the resort to notions which are not constructive but only *definite*, perhaps in the sense of being expressible in the Kleene hierarchy. An evaluation of the advantage over formalism depends of course on one's answer to these questions: Is an interpretation necessary or even desirable? How "natural" is the given interpretation in terms of operations on symbol strings? What alternative interpretations are available? The interpretation given here seems most convincing in the case of set theory and especially of analysis, where it seems hardly less "natural" than the already "artificial" arithmetized analysis and indeed may be regarded as a further development of it. One should like to know therefore whether, without sacrificing its empirical spirit, one can modify this interpretation of set theory and then wed it to a more common interpretation of elementary logic. (On the other hand, one may also wish to develop a constructive formal system of strata as a strengthening of the ramified theory of types, and then look for a more traditional interpretation of it.)

The book may lead to results outside the field of foundations, since it is often convenient to adjoin "purely symbolic" operations to operations that can be interpreted. Admissibility is then a key concept. However, no results have been produced so far. Thus, while the similarity up to now of operative to ordinary analysis is gratifying, it is also a little disappointing.

The main weakness of the book is its failure to indicate as clearly as a work on foundations should the strength of the methods employed and thus of the underlying assumptions. Claims that only the principles of the chapter on *Protologic* are employed and that no understanding of other logical notions is required are unconvincing, and seem unnecessary. Understanding the notion of an improper system, e.g., requires more than ability to check or even to anticipate the use of rules. A discussion would have been helpful of the extent to which the arguments used could become subjects of direct consistency proofs. This would have shown more clearly the relationship to Hilbert's program, the places where transfinite induction is used, and the places where formalization seems impossible. Arguments concerning constructive *eliminability* seem in general more elementary in nature than those concerning admissibility. Hence a clearer description of this notion for metarules, metametarules, . . . and a clearer labelling of those cases where actual eliminability has been proved would have been desirable. Failure to maintain the distinction between eliminability and admissibility also makes the discussion of the law of excluded middle and of *stability* (§8, §16) puzzling, since $\neg\neg A \rightarrow A$ seems always admissible though not always eliminable.

Other shortcomings of the exposition are the following: It is often too sketchy or condensed to allow careful checking. Examples are §10 and lack of a careful definition of *predecessor*, needed for *fundiert* which is critical for the notion of induction scheme. Other distinctions besides that between admissibility and eliminability are not maintained rigorously, e.g., that between rules, metarules, . . . and their forms (§6) or that between admissibility and derivability (§8, §9). The underlying system K, and indeed the fact that admissibility in K is being examined, is not always clearly indicated. In connection with relative admissibility (§7), two major though probably corrigible errors occur. First, as H. B. Curry discovered, the rule a, $b \rightarrow a \wedge b$ is not relative admissible, e.g., in the system whose only rules are $+$ and $+a \rightarrow 0$. Second, $a \wedge b \rightarrow a$ is not admissible, e.g., in the system whose only rules are $+$ and $+a+ \rightarrow ++a+$ and (D_1^0). Despite all these shortcomings, which probably require a thorough revision to overcome, the book constitutes an essentially sound and indeed outstanding contribution.

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Introduction to Mathematical Logic, Vol. I, by Alonzo Church. Princeton, The Princeton University Press, 1956. 10+376 pp. $7.50.

This is a revised edition of the slim, paper-backed volume which appeared in 1944 as one of the Annals of Mathematics Studies. Of the five chapters two are devoted to propositional calculus, two to functional calculi of first order, and the last to second-order calculi, so that the plan of the original edition is followed to treat the most basic formal systems of mathematical logic. But the material of the original has been so greatly altered and expanded that it seems best to report the important features of the new work directly, rather than to compare it in detail with its predecessor.

Several unusual features of the book are apparent even before one begins to read: There is an introduction 68 pages in length; it is divided into 10 sections, which is exactly the number of sections in each of the five chapters; there are 590 consecutively numbered footnotes, at least one of which is a full page in length; there was a lapse of five years between the completion of the writing and the appearance of the published volume. While there are separate sections for historical notes, these spill over liberally into the highly informative footnotes; so that when the author's pre-eminent reputation for painstaking attention to historical detail is considered, it seems manifest that this work will quickly establish itself as a definitive reference volume. The book also contains a great many exercises, ranging from simple illustrations to brief sketches of developments not treated in the text. For some curious reason textbooks in symbolic logic have always evinced a conspicuous paucity of problems suitably challenging to mathematically inclined students, a phenomenon which has tended to place this subject at a competitive disadvantage with most of the other mathematical sciences. The present innovation is thus very welcome, and will greatly enhance the value of the book's use in connection with beginning courses for students with some background of mathematical experience.