Curvature and Betti numbers. By K. Yano and S. Bochner. (Annals of Mathematics Studies, no. 32.) Princeton University Press, 1953. 10 and 3–190 pp. $3.00.

This book is a unified and systematic account of a number of contributions to differential geometry in the large which have appeared in papers of Bochner over the past several years and in some recent papers of Yano, Lichnerowicz, and others. The well-spring of all these results is the theorem of E. Hopf-Bochner (an application by Bochner of a theorem of E. Hopf) which states that if ϕ, a function defined on a compact Riemannian manifold, has positive (negative) semidefinite Laplacian, i.e., using semicolons to denote covariant differentiation, $\Delta\phi \equiv g^{ij}\phi_{;i;j} \geqq 0$ $(\leqq 0)$, then $\Delta\phi$ vanishes and ϕ is a constant. Then this is applied to obtain necessary conditions for the existence of various types of tensor fields by considering $\Delta\phi$ for the function $\phi = \xi^{i_1\cdots i_p}\xi_{i_1\cdots i_p}$, where $\xi^{i_1\cdots i_p}$ are the contravariant components and $\xi_{i_1\cdots i_p}$ the covariant components of the same tensor, and deriving conditions under which $\Delta\phi \geqq 0$ and $\phi = 0$. Although this in barest outline is the basic approach, in each application a study of the expression $\Delta\phi$ is required, and many technical difficulties arise in the search for conditions on the metric sufficient to insure definiteness of $\Delta\phi$ for various classes of tensors. Since $\Delta\phi$ involves the second covariant derivatives, interchange of the order of differentiation via the Ricci formula introduces the curvature tensor. Thus for example in the first case treated by the authors, that of vector fields ξ_i, they obtain easily

$$\text{(i)} \qquad \frac{1}{2}\Delta\phi = \xi^{i;j}\xi_{i;j} + R_{ij}\xi^i\xi^j$$

if ξ_i is a harmonic vector field, i.e., if $\xi^i{}_{;i} = 0 = \xi_{i;j} - \xi_{j;i}$, and

$$\text{(ii)} \qquad \frac{1}{2}\Delta\phi = \xi^{i;j}\xi_{i;j} - R_{ij}\xi^i\xi^j$$

if ξ_i is a Killing vector field, i.e., if $\xi_{i;j} + \xi_{j;i} = 0$. From (i) follows Myers' theorem that the first Betti number of a compact manifold of positive Ricci curvature is zero, and from (ii) follows that there exists no one-parameter group of isometries on a compact manifold of negative Ricci curvature. These results, although for the simplest case, are quite typical. Thus in the following chapters tensors of higher order are considered and more complicated statements generalizing (i) and (ii) are obtained. It is not always possible to give simple criteria for the definiteness of $\Delta\phi$, but the special cases of flat manifolds and manifolds of constant curvature are considered and results on the existence of harmonic and Killing tensors obtained. Several measures of deviation from constancy of curvature are defined and it is shown that if this deviation remains within certain limits the Betti numbers are unaltered. To the reviewer these relations between curvature and Betti numbers are among the most interesting consequences of the theory.

A chapter is devoted to the special case of semi-simple group manifolds, and here the deviation from flatness is explicitly calculated. Following this is a chapter on Riemannian manifolds carrying additional structure in the form of an affine connection with torsion (as in the case of the group space). A chapter on Kähler manifolds completes the main body of the book. This latter chapter includes important applications of the theory; for example, it is shown that if the deviation of the curvature of a Kähler manifold from constant positive holomorphic curvature remains within prescribed limits, then there are no effective harmonic tensors, and hence the Betti numbers are those of the complex projective space.

A final chapter by S. Bochner entitled "Supplements" is perhaps the most significant since it contains indications of new directions in which the theory is proceeding. Important as it is, however, the topics are so diverse as to make a brief summary impossible. The titles of the eight sections are: (1) Symmetric Manifolds, (2) Convexity, (3) Minimal Varieties, (4) Complex Imbedding, (5) Sufficiently Many Vector or Tensor Fields, (6) Euler-Poincaré Characteristic, (7) Noncompact Manifolds and Boundary Values Zero, (8) Almost Automorphic Vector and Tensor Fields.

The book contains in addition to the above a chapter outlining the relevant differential geometry and tensor analysis and a brief introduction to complex-analytic manifolds. It should be remarked that in addition to interesting theoretical contributions, Yano should be commended for the careful, readable exposition he has given here of this topic in global differential geometry.

WILLIAM M. BOOTHBY

Higher transcendental functions. By A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi. Based, in part, on notes left by Harry Bateman and compiled by the Staff of the Bateman Manuscript Project. New York, McGraw-Hill, 1953. Vol. I, 26+302 pp., \$6.50. Vol. II, 17+396 pp., \$7.50.

These two volumes compiled by the "Bateman Manuscript Project" represent a stupendous accomplishment. Under the able direc-