extended complex plane such that for all rational functions $u(z)$ satisfying $|u(z)| \leqq 1$ on Z, the transformation $u(T)$ exists and $\|u(T)\| \leqq 1$. It is proved that unitary and symmetric transformations are characterized by the fact that they have respectively the unit circle and the real axis as spectral sets.

E. R. LORCH

Über die Klassezahl abelscher Zahlkörper. By Helmut Hasse. Berlin, Akademie, 1952.

This is a highly technical book, whose object is the derivation of a formula for the class-number h of an arbitrary absolute abelian field K and the study of this formula. Such a formula had been proved by Kummer for cyclotomic fields (i.e. fields generated by a root of unity) and in the general case by several authors (Fuchs, Beeger, Gut). The source of these formulae is of course the fact that the class number appears in the expression of the residue at $s=1$ of the zeta function $\zeta_K(s)$ of K. Using the product decomposition of ζ_K into L-series, one is reduced to the computation of the values $L(1;\chi)$ at 1 of the L-series corresponding to those characters $\chi\neq 1$ which are associated to K by class field theory. The numbers $L(1;\chi)$ appear as infinite series; the main problem is to express them in closed form, which is done by making use of Gaussian sums.

The resulting formula appears in the form $h=h_0h^*$, where h_0 is the class-number of the maximal real subfield K_0 of K, while h^*, the "second factor" of h, turns out to be an integer >0. The fact that these two factors h_0 and h^* are actually integers is not obvious from the expressions for these numbers which appear in the formula itself. One of the aims of the author is to transform these expressions in such a way as to render their arithmetic nature more apparent. This in itself would not appear so very fascinating a task: when we express the number of zeros of an analytic function in a region by a contour integral, we do not take pains to establish independently that the value of this integral is an integer. However, in the process of so doing, new properties of h_0 and h^* appear which lead to a certain number of new results on class numbers of fields.

The second chapter of the book is concerned with the transformation of the expression for h_0. Here the striving to obtain for h_0 an expression which exhibits it as an integer is not entirely successful. Two different lines of attack are followed which yield results for two different kinds of fields K. The end results of the two methods are in the following form: the product of h by some integer c is expressed as the index in the group of all units of a certain sub-group generated by explicitly given units. The number c is 1 for certain categories of fields. As a consequence of the formulas he obtains, the author obtains generalizations of Weber's theorem to the effect that the largest real subfield of the field of 2^pth roots of unity has an odd class number; for instance, h_0 is odd if K_0 is cyclic of degree 2^r and if its conductor is a power of a prime.

The third chapter is concerned with the second factor h^*, which occurs of course only in the case where K is imaginary. It is first proved by class-field theory that h^* must be an integer. This being done, the author concentrates on the study of the expression of h^*. One of the factors which occur in it is the index in the group of units of K of the subgroup generated by the units of K_0 and by the roots of unity of K. This index Q is always 1 or 2, but whether it is 1 or 2 cannot be ascertained by the methods of class field theory alone: this is a question which involves some non-abelian features. The author gives various criteria to compute Q. Next, he succeeds in the case of h^* in giving an expression of this number which exhibits it as an integer. As a result, the author obtains the following generalizations of Weber's theorem: if the conductor of K is a power of 2, then h is odd—if the conductor of K is a power of 3, then h^* is not divisible by 3. It is known since Kummer that, for the field of pth roots of unity, p a prime >2, then h can only be divisible by p or by 2 when h^* itself is. It is proved that the part of this statement concerning the divisibility by 2 extends to the case of arbitrary cyclic imaginary fields K; moreover, necessary and sufficient conditions for h to be odd are given in that case.

The book concludes with a table of values of h^* for all fields of conductors <100, and of some other quantities relative to these fields.

The amateur of special concrete arithmetical facts will be highly gratified by the reading of this book, which abounds in various examples. However, it must be admitted that the extraordinary ingenuity displayed by the author in the derivation of these results can only confirm the impression that class numbers behave in a rather chaotic fashion or, at any rate, that if they are governed by general laws, then we have as yet no inkling of what these laws may be like.

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