rank 2. In the former case the singularity is called a center, if the form $\sum_{i,j=1}^{n} a_{ij}x^i x^j$ is positive definite. When V_n is compact, with $n \geqq 3$, a completely integrable field E_{n-1} with a nonempty set of centers as singularities must have leaves which are homeomorphic to an $(n-1)$-sphere. Moreover, this happens only when there are exactly two centers and when V_n is homeomorphic to an n-dimensional sphere. The chapter is concluded by a more detailed study of the case when the form is analytic.

There does not seem to be any doubt to the reviewer that both studies contain valuable contributions to the topology and differential geometry of manifolds. We also believe that they only mark a beginning of further fruitful investigations.

SHIING-SHEN CHERN

Grundzüge der theoretischen Logik. By D. Hilbert and W. Ackermann. 3d ed. Berlin, Springer, 1949. 8+155 pp.

Principles of mathematical logic. By D. Hilbert and W. Ackermann. Trans. by G. G. Leckie and F. Steinhardt; ed. and with notes by R. E. Luce. New York, Chelsea, 1950. 12+172 pp.

The first edition of this book appeared in 1928. According to the preface, it was based on Hilbert's lectures of 1917–22. It was reviewed, somewhat unsympathetically, by Langford in this Bulletin, Vol. 36, pp. 22 ff. The second edition appeared in 1938; it was reviewed by Rosser in this Bulletin, Vol. 44, p. 474, and by Quine in Journal of Symbolic Logic, Vol. 3, p. 83. The third edition and the English translation of the second edition, with both of which this review is concerned, appeared almost simultaneously in 1949–50. They have been previously reviewed in the Journal of Symbolic Logic, Vol. 15, p. 59, by Church, and Vol. 16, p. 52, by Zubieta, respectively.

The book was intended as an introductory textbook of mathematical logic in a narrow sense. The Hilbert school never subscribed to the identification of mathematics and logic, and regarded "mathematical logic," "theoretical logic," and "logical calculus" as synonymous designations for a preliminary stage in the subject of "foundations of mathematics," which many Americans prefer to call "mathematical logic" in a broader sense (cf. Quine's book of 1940). Anything depending on an axiom of infinity or similar assumption would belong to the latter subject but not to the former. Nevertheless, the authors regard the narrower subject as an essential step to the broader. Thus, in the first preface, signed by Hilbert, it is stated that the book is intended as a preparation for a further book by him and Bernays, no doubt referring to the great work of which the first volume appeared in 1934.

The plan of the book has remained the same in all the editions. There are four chapters, dealing respectively with propositional algebra, the algebra of classes or monadic predicates, the restricted calculus of predicates, and various extended calculuses of predicates. The first and third chapters begin with an intuitive discussion, introduce the formalization gradually and naturally, and conclude with a formal system and its properties of deducibility (consistency, independence, completeness, decidability, etc.). This plan is modified in the second and fourth chapters, which are briefer and less formal. The second chapter is something of a side issue; it treats the class interpretation of Boolean algebra, to which has been added an operation of universality, so as to form a modal algebra (essentially the Lewis system S5) adequate for one formulation of the traditional syllogism. The fourth chapter is more of a survey; but it gives a good idea of the intolerable complexity of the theory of types.

The changes made in the various editions have, in fact, been relatively minor. For those made in the second edition, see the previous reviews. In the third edition some changes occur in the second, third, and fourth chapters. In the second chapter the second edition contains an intuitive treatment with a reference, for all details, to a paper by Wajsberg; in the third edition the intuitive discussion is somewhat amplified, there is a decision process on an intuitive basis for the case where one modal operation does not include another, and there is no reference whatever to Wajsberg. In the third chapter certain technical errors are corrected (see Church review), and new results on the decision problem are incorporated. In the fourth chapter there is an amplified and more accurate treatment of the theory of types. The second edition contained a brief but useful bibliography; the third contains only footnote references in the text.

As a text the book has become a classic. Such terms as "conjunctive normal form" and "restricted predicate [originally 'functional'] calculus" have become well known largely through its influence. Despite certain shortcomings it is still the best introduction for the student who seriously wants to master the technique. Some of the features which give it this status are as follows:

The first feature is its extraordinary lucidity. A second is the intuitive approach, with the introduction of formalization only after a full discussion of motivation. Again, the argument is rigorous and exact with full attention to the difference between a rule and an axiom. A fourth feature is the emphasis on general extra-formal principles (in more recent terminology epitheorems or, in some cases, metatheorems) such as normal forms, replacement rules, decision procedures, etc. Finally, the work is relatively free from bias. In its origin it was a judicious combination of the algebraic methods of Schröder with the logistic standpoint of Frege and Russell. (Thus, the emphasis on propositional interpretation agrees with the latter, while the conjunctive normal form is essentially Schröder's "Entwicklung." There are also notable analogies with the Post paper of 1921 (probably because both drew from the same sources). It therefore represents a union of the two principal tendencies of the time. Since the different schools of mathematical thought differ not so much in regard to logic (in the narrow sense) as in regard to its relation to the foundations of mathematics, the book represents a rather complete treatment of the heart of its subject, acceptable from any point of view. All together, the book still bears the stamp of the genius of one of the great mathematicians of modern times.

On the other hand, there are signs of obsolescence. Much water has spilled over the dam in the twenty-five years since this book was first published. This book is confined entirely to the classical logic; whereas we now know there are a variety of generalized systems, some with interesting applications, which have much the same relation to classical logic that various generalized geometries do to Euclidean. The new techniques developed by Gentzen are barely mentioned. Of course, it is quite true, as some insist, that one cannot do anything with those techniques that one cannot do with the older ones; just as one can go anywhere with a horse and buggy that one can reach with a Cadillac. The mere mentioning of such techniques is not enough; mathematical logic is now capable of an approach which is as different from that in this book as the latter was from its predecessors. Moreover, the separation between logic proper and mathematics can no longer be maintained. Recent foundational studies (recursive arithmetic, combinatory logic including the theories of lambda conversion, Post's formalized syntax, etc.) show that important theories can be constructed without the aid of any logical calculus, and that these are sufficient for portions of mathematics; so that logic is founded on mathematics, as the intuitionists have long held, rather than the reverse. An elementary text along these lines has not yet appeared; but it is certainly possible, and it is to be hoped that it will not be long delayed. Until it appears the present text will lack the completeness that it had when it was first issued.

So much for the German book. Let us turn now to the English translation. This was made from the second German edition without the consent, or even the knowledge, of the surviving author. (This situation has, incidentally, caused some unfavorable comment which the reviewer heard during his visit to Europe in 1950–51.) On the whole the translation is a good one; the reviewer thinks that the translators have succeeded rather well in their stated aim, "both to give an exact English rendering of the sense and intent of the original text and also, so far as possible in a different language, to reproduce something of its manner and style." There are to be sure a number of what the reviewer considers mistranslations, such as "number" for "Anzahl," (instead of "cardinal number,"—"number" is the translation of "Zahl"), "sentence" for "Aussage," and "expression" for "Satz"—the reviewer's objections to the second of these will be stated below—and (cf. review by Zubieta) "Ersetzung" should be translated "replacement" rather than "substitution" (=Einsetzung). The translators and/or editor have also corrected some technical inaccuracies in the text of Chapter 3, essentially the same as those made in the third edition (see Zubieta's review), have added a few pages of "editor's notes," and have revised the bibliography and index.

It remains to comment on the use of "sentence" as a translation for "Aussage" (and of certain related words correspondingly). The editor and translators are evidently quite conscious that this changes the sense of the original and go to some length to justify it as a "correction" in the editor's preface and notes. It is evidently due to the fact that they believe that in logic the objects of study are linguistic symbols and expressions. The reviewer's view is that in logic it is irrelevant what the objects of study actually are (so long as they satisfy the formal conditions). If one chooses to regard these objects as symbolic, one is at liberty to do so; but in that case one must distinguish symbols used (U-symbols) and those mentioned (O-symbols). If the O-symbols can also be used, then a quite elaborate machinery must be employed to maintain the distinction between use and mention. All this bewildering machinery can be avoided without entailing confusion if one does one of two things: (a) adopt O-symbols which are actually meaningless, so that if one occurs in a context it can designate only itself; or (b) avoid specifying that the objects are linguistic, adopting a terminology which does not suggest that eventuality. In either case all symbols are U-symbols, and the ordinary conventions of good linguistic usage suffice to avoid confusion. Hilbert preferred (a), and the book under review does also; but the preference is not insisted upon, and the natural connotation of "Aussage" leaves the reader free to adopt (b). (This is part of that freedom from bias which was previously commented on. In either case there is no "recurrent carelessness in maintaining a strict distinction" between use and mention, as charged in the editor's preface.) The English word "proposition" has the same connotation. Moreover, it is not necessary, in using it, to settle the question, mentioned in the editor's Note 1, as to what propositions are. But the word "sentence," particularly when accompanied by such a preface, suggests an insistence on the linguistic viewpoint. This imposes on the work a bias which was not there originally; and it interferes somewhat with the intuitiveness of the approach also. If, as the editor himself comments in his Note 1, it is not necessary to decide the nature of the objects of logical study, why insist on a terminology which commits one to a particular view of it? It is a pity that what is otherwise an excellent translation should be marred by such pedantry.

To sum up, the book of Hilbert and Ackermann is one of the classics of the logical literature. In spite of the fact that it reflects the state of the science twenty-five years ago, with some changes of detail but no thorough-going revision, it is still the best textbook in a Western European language for a student wishing a fairly thorough treatment. The English translation is well done. The reviewer regrets certain features of it, and in particular regrets that it was published without consultation of the surviving author; but that does not alter the fact that it is a real service to English speaking students of the subject. Its content does not differ from that of the latest German edition in any important way.

HASKELL B. CURRY

Introduction to the theory of games. By J. C. C. McKinsey. New York, McGraw-Hill, 1952. 10+371 pp. $6.50.

This book is intended as a textbook for advanced undergraduate and graduate students. It fills, perhaps uniquely at present, a wide existing need which the now classical book of von Neumann and Morgenstern cannot satisfy on this level. In addition to the normal interest of mathematicians in the theory of games there is also the great interest of economists and many applied mathematicians in the theory; much of what is now called operations analysis, military and otherwise, makes copious use of this theory. This textbook, therefore, which presupposes essentially a knowledge of advanced calculus, will be useful to students and workers in several fields. In the scope of about 360 pages it discusses the principal topics which, by general agreement, should fit into an introductory text. Most of the book is devoted to zero-sum two-person games, but there are several chapters