BOOK REVIEWS

Intégration (Chap. I–IV). By N. Bourbaki. (Actualités Scientifiques et Industrielles, no. 1175.) Paris, Hermann, 1952. 2+237+5 pp. 2500 fr.

The volume under review contains the first four chapters of the sixth book of the first part of the series that the authors started publishing in 1939. One difficulty in the way of a complete critical evaluation of the volume is that at several points the proofs are made to depend on the results of the fifth book (entitled, according to the advertisements, *Espaces vectoriels topologiques*), and that, at the time this is being written, none of the chapters of the fifth book has appeared yet. Another such chronological difficulty is that the volume is only a half (or less) of the entire projected book; even the world's greatest art expert might hestitate to pronounce judgment on Picasso's latest if he were shown only the bottom half of it. Laboring under these two handicaps, I shall do the best I can to give the reader of this review an idea of what he may expect from Bourbaki's treatment of integration.

Mathematics books may be roughly classified as either textbooks or monographs. There is, of course, no clear line between the two classes, but certain tendencies are generally visible. The author of a textbook is mindful of the troubles of beginners; he tries to provide a systematic introduction to his subject; he is prepared to include a liberal supply of exercises; and, faced with an upper bound on the number of pages in his book, he is willing to barter completeness for clarity. The author of a monograph, on the other hand, addresses himself to the expert; he starts where his predecessors left off; he is not bound to mention even the important applications of the theory; and, faced with an upper bound on the number of pages in his book, he will prefer condensed proofs to the charge of bibliographic highhandedness. The famous *Ergebnisse* series, for example, consists of monographs; the volume under review is a textbook.

The expert in modern integration theory will certainly not find anything to startle him here, except possibly the order of the ideas and some of the terminology used to describe them. Hölder's and Minkowski's inequalities, vector lattices, measures on compact and locally compact spaces, vector-valued integrals, product spaces, outer measure, functions defined almost everywhere, the L^p spaces, approximation by step functions, measurable functions and measurable sets —this is a representative selection of the items listed in the table of contents (in that order), and certainly they are all of the sort one expects to find in a book on integration. No previous knowledge of integration or measure theory is assumed. The important and classical instances of the theory (Lebesgue measure in Euclidean space and measures in discrete spaces) receive some special attention. As for the bibliography, it contains exactly three items: an article by Hölder in the *Göttinger Nachrichten* (1889), the second edition of Minkowski's *Geometrie der Zahlen* (1910), and *Inequalities* by Hardy, Littlewood, and Pólya (1934).

In the universal preface (*Mode d'emploi de ce traité*), a copy of which is included with every volume of the series, the authors make it plain that they are writing for beginners and that the series as a whole is, in principle, self-contained. They promise to provide exercises, and they propose to give only those references to the literature that are likely to be of profit to the reader, omitting those that serve only to establish priority.[1]

There is no point in belaboring the issue; we are dealing with a publication that by intention and *de facto* is a textbook. It must be judged as a textbook. Putting ourselves in the place of a student, we must ask: "Is the subject important, is the book clearly written, and is the material well organized?" Putting ourselves in the place of the supervisor of a Ph.D. thesis in one of the applications of integration (e.g., ergodic theory, probability theory, length and area, Boolean algebra, or integral geometry), we must ask: "Is this the point of view that will help a student to understand and to extend his field of interest?" I say that the answer to the student's question is yes and the answer to the professor's question is no.

It is, to be sure, a very special kind of student who can read this book. The prospective reader has to know substantial portions of

[1] In passing, one might ask whether the listing of Hölder and Minkowski is consistent with this announced policy. There are three other points in the preface that cannot be said to be in perfect agreement with subsequent developments. (1) "Les références seront groupées dans un exposé historique, placé le plus souvent à la fin de chaque chapitre et où l'on trouvera, le cas échéant, des indications sur les problèmes non resolus de la théorie." Regrettably, there is only one historical note in this volume and that one discusses nothing but the inequalities of Hölder and Minkowski; there are no indications of unsolved problems. (2) "On s'est efforcé de ne jamais s'écarter de la terminologie reçue sans de très sérieuses raisons." The authors' term for a *vector lattice* is *Riesz space*. What Hausdorff called a *field* (of sets), and what I more recently called a (Boolean) *ring* (of sets) is called a *clan*. (3) " . . . les nécessités de la démonstration exigent que les chapitres, les livres et les parties se suivent dans un ordre logique rigoureusement fixé." Apparent conclusion: one must know the separation theorems for convex sets in topological vector spaces before one can find out what the Lebesgue measure of the closed unit interval is.

Bourbaki's version of general topology (e.g., filters and uniform structures) and at least a modicum of Bourbaki's version of algebra (e.g., tensor products and ordered groups). Moreover, he has to be at home in topological vector spaces; at the very least, he has to be familiar with pseudo-norms and he must know the Hahn-Banach theorem. If, on top of all that, he has sympathy for the axiomatic and abstract approach to mathematics (this must surely be a consequence of the preceding conditions), then he can proceed to read, to understand, and to enjoy himself.

The basic philosophy of the authors is that the proper context for integration theory is a locally compact Hausdorff space. Three arguments are advanced to support this view. (1) The generality in considering "abstract" measures is illusory, since (via the Stone theory of representations of Boolean algebras) every such measure is isomorphic, in some sense, to a measure on a locally compact space. (2) In the great majority of applications, the spaces that arise come equipped with a locally compact topology. (3) In the few instances where no (locally compact) topology is apparent, it is often useful to introduce one.

I should like to call attention to the fact that there is another side to the ledger. (1) It often happens (e.g., in ergodic theory) that the consideration of the Stone representation space is of no help at all. There is, for instance, a theorem of von Neumann (Annals of Mathematics, 1932), the object of which is to conclude that, under suitable hypotheses, every automorphism of the reduced Boolean algebra of a measure space is induced by a measure-preserving transformation of the space itself. The corresponding assertion for a Stone space is trivial, and, in this context, misses the point. (2) While it is true that discrete spaces, Euclidean spaces, and locally compact groups are locally compact, that fact, as an argument for restricting the study of measure theory to locally compact spaces, is a double-edged sword; it argues just as well for restricting the study of topology itself to locally compact spaces. It argues also against measure theory in, say, general metric spaces; the theory of Hausdorff measure is thereby deprived of a right to existence. (3) One of the most fruitful applications of measure theory is the modern theory of probability. In that theory a locally compact topology virtually never comes up. The topologies that workers in stochastic processes occasionally do meet are likely to be (a) not locally compact and (b) not useful.

The authors themselves are apparently not completely of one mind on the issue. My principal objective evidence for this assertion is that five of the six sections into which Chapter IV is divided include, at the end, a few pages (in small print) devoted to the case of abstract measures. If those pages are meant to pacify the large body of analysts who learned measure theory from, for instance, the classic book of Saks, the desired effect will probably not be achieved. Instead of recognizing an old friend in the pages on abstract measure, the analyst will find that the theory there expounded goes back no farther than four years; in fact, that theory is substantially identical with the theory presented in Stone's four notes on integration (Proc. Nat. Acad. Sci. U. S. A., 1948 and 1949).

The first two (short) chapters of the book are devoted to inequalities and vector lattices, respectively. A very general form of the Hölder and Minkowski inequalities (proof based on the Hahn-Banach theorem) is derived in Chapter I; the subsequent (integral) versions of these inequalities are in turn based on the general ones. Chapter II treats the elementary theory of vector lattices; subspaces, product spaces, and positive linear functionals receive their due share of attention.

The main work begins in Chapter III. A measure (on a locally compact space E) is defined as a linear functional μ on the space $\mathcal{K}(E)$ of continuous functions with compact support, subject to the condition that, for every compact set K, the restriction of μ to the functions whose support is contained in K be continuous with respect to the topology of uniform convergence. An alternative notation for the value of the measure μ at the function f (in addition, that is, to the obvious notation $\mu(f)$) is $\int f(x)d\mu(x)$. The justification for this unusual procedure is, of course, the general Riesz theorem, which asserts that there is a one-to-one correspondence between (regular Borel) measures ν in the classical sense and measures μ in the Bourbaki sense, such that μ and ν correspond to each other if and only if $\mu(f) = \int f(x)d\nu(x)$ for all f. The Riesz theorem, however, is not an essential part of the authors' theory; a somewhat disguised version of it appears in the latter part of Chapter IV (p. 165), in small print.[2] (Some traces of an ambivalent attitude are again visible in connection with the definition of measure. The motivation for studying integration theory, as presented in the introduction, is based on concepts such as length, area, volume, and weight; expressions such as "a unit mass placed at a point" occur throughout the book.) After introducing the basic concepts (e.g., positive measure, bounded measure, the norm and the support of a measure) and some of the relations among them,

[2] The Riesz theorem for Stieltjes integrals on the line is mentioned in the introduction; no relation between it and the theorem on p. 165 is ever indicated.

Chapter III concludes with a discussion of integrals of vector-valued functions and measures on product spaces.

Chapter IV is the longest one; it contains more than half of the total number of pages. It begins with the definition of the upper integral $\mu^*(f)$ of a positive, lower semi-continuous function f with respect to a positive measure μ (approximation from below by continuous functions). The authors' insistence on avoiding functions of sets whenever possible, and using functions of functions instead, is manifested in the importance of lower semi-continuous functions as a technical device; the point is, of course, that the characteristic function of a set A is lower semi-continuous if and only if A is open, so that lower semicontinuous functions in this theory play the role of open sets. The extension procedure is continued one more step: approximation from above by lower semi-continuous functions yields the upper integral of an arbitrary positive function. The outer measure of a set is defined as the upper integral of its characteristic function. The second section is devoted to "almost everywhere" properties. (A set A is called *μ-negligible* if $\mu^*(A)=0$.) In the third section the L^p spaces are defined, it is proved that they are complete, and the Lebesgue dominated convergence theorem is derived. The fourth section contains various standard results about integrable functions (e.g., the monotone convergence theorem and approximation by set functions). The fifth section studies the concept of measurability and the sixth (and last) section returns to the study of the fundamental inequalities.

Owing, no doubt, to the authors' predilection for using as definiens what for most mathematicians is the definiendum, there are many spots at which the treatment appears artificial. Where others can say immediately that a point x belongs to the support of a measure μ if and only if $\mu(V)\neq 0$ for every neighborhood V of x, the authors are first forced to demand, for every neighborhood V of x, the existence of a continuous function f whose compact support is contained in V and for which $\mu(f)\neq 0$. After defining the space $\mathcal{F}^p$ of all functions f for which the upper integral of $|f|^p$ is finite, the authors define $\mathcal{L}^p$ as the closure in $\mathcal{F}^p$ of the set of continuous functions with compact support—an effective but startling method of enforcing measurability. Some mathematicians might regard as artificial a treatment in which the evaluation of the Lebesgue measure of the unit interval (p. 156) is made to depend on what is essentially the Riesz-Fischer theorem ($\mathcal{F}^p$ is complete, p. 130). The important concept of *clan* (Boolean ring) is defined as follows: a non-empty class Φ of subsets of a set A is a clan if there exists a real algebra $\mathcal{A}$ of numerical functions on A, such that the sets in Φ are exactly the ones whose characteristic function belongs to $\mathcal{A}$. (In order to make the concept usable, it is, of course, necessary that the definition be immediately followed by the theorem: Φ is a clan if and only if it is closed under the formation of unions and relative complements.)

The last example of artificiality that I shall adduce deserves a paragraph to itself. Measurability is the last important concept defined in the volume (p. 180); I quote the definition verbatim. "*Soient E un espace localement compact, μ une mesure positive sur E. On dit qu'une application f de E dans un espace topologique F est mesurable pour la mesure μ* (ou encore *μ-mesurable*) *si, pour toute partie compacte K de E, il existe un ensemble μ-négligeable $N \subset K$ et une partition de $K \cap \complement N$ formée d'une suite* (*finie ou infinie*) (K_n) *d'ensembles compacts, tels que la restriction de f à chacun des K_n soit continue.*" Three comments should be made. First, the history of the definition should be obvious to anyone who has studied measure theory elsewhere; we are looking at Lusin's theorem. Second, the apparent purpose of the definition is to minimize the role of set-algebra in integration theory; although subsequently we must be told of the characterization in terms of inverse images (e.g., the proof that a lower semi-continuous function is measurable depends on that characterization), the original appearance of the concept is doggedly kept on the functional level. Third, the effect of the definition is to help perpetuate the myth that the measurability of a function can only be defined by reference to a measure—a myth quite as unfounded as the (fortunately moribund) myth that the continuity of a function can only be defined by reference to a metric. Both concepts are set-theoretic, and the similarity between them is worth emphasizing.

Some minor technical points should be mentioned. The distinction between $\mathcal{L}^p$ (the function space) and L^p (the quotient space of equivalence classes) is rigorously maintained; the authors effectively demonstrate that that can be done without getting hopelessly mired in notation. The section (pp. 79–89) on weak vector integration is excellent. It is clear and helpful and puts into evidence the essential role played by the condition: "the closed convex hull of a compact set is compact." *In re* terminology: "integrable set" seems to me to be a good phrase (better than "measurable set of finite measure"); on the other hand, "completely additive" instead of "countably additive" is both unfashionable and misleading. There is a slight confusion about the notation for double integrals. On p. 42 it is asserted that it is sometimes desirable to use a symbol such as $\int\int f(x)d\mu(x)$. What is probably meant is that the "double" integral $\int f(x, y)d\nu(x, y)$, where ν is the product measure of, say, λ and μ, is alternatively denoted by $\int\int f(x, y)d\lambda(x)d\mu(y)$; the corresponding iterated integrals are denoted by symbols such as $\int d\lambda(x)\int f(x, y)d\mu(y)$. The threat to use two integral signs with only one differential is never actually carried out. Aside from this triviality, I noticed no errors in the book, trivial or otherwise. I caught only three minor misprints; the only one that might worry a student for a few seconds is on p. 121, line 9: f_n should be $\tilde{f}_n$.

The topics centering around the names of Fubini (decomposition of measures), Radon and Nikodym (absolute continuity), and Haar (group invariance), are not discussed in this volume; according to the introduction, they will be treated in the subsequent chapters.[3] The introduction indicates also that the authors are planning eventually to apply their theory (probability) and to generalize it (distributions).

My conclusion on the evidence so far at hand is that the authors have performed a tremendous *tour de force*; I am inclined to doubt whether their point of view will have a lasting influence.

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Introduction to modern prime number theory. By T. Estermann. Cambridge University Press, 1952. 10+75 pp. $2.50.

The main purpose of this tract is "to enable those mathematicians who are not specialists in the theory of numbers to learn some of its non-elementary results and methods without too great an effort." Actually, the book is devoted to the limited object of proving the Vinogradov-Goldbach theorem that every sufficiently large odd number is the sum of three primes; in the course of proving this result, the author supplies the necessary results on characters and primes in arithmetic progressions. Only a few elementary number-theoretic results are assumed, these being quoted from Hardy and Wright's book *An introduction to the theory of numbers*; Cauchy's residue theorem is also assumed.

The book is a very carefully thought out exposition which lays bare the whole nature of the proof and unremittingly avoids all things not needed in the final proof. This leads to a work which is somewhat austere although not so formal as Landau's *Vorlesungen über Zahlentheorie*; in common with Landau's book, Estermann's tract gives few references to the literature. Nevertheless, the author admirably succeeds in his aim. The proofs are clear and remarkably

[3] In Chapter III, the sole assertion resembling Fubini's theorem is stated for continuous functions with compact support only.