BOOK REVIEWS

Arithmetical questions on algebraic varieties. By B. Segre. London, Athlone, 1951. 5+55 pp. 10s. 6d.

This book is based on three lectures given by the author in King's College, London, in March 1950. He describes it in the preface as "an account of the more important problems in which algebraic concepts and methods have proved fundamental in overcoming arithmetical difficulties, or where arithmetical notions and results have played a rôle in algebraic geometry." The exposition is throughout very condensed, as might be expected on comparing the small size of the book with the large field covered. Results are quoted largely without proof, or with only an outline of the ideas used in the proof; there is however an excellent and extensive bibliography, by means of which the reader attracted by any of the subjects touched on can have no difficulty in informing himself more fully.

The first chapter, presumably corresponding to the first lecture, deals with quadrics in an arbitrary commutative field γ. First is considered the rather exceptional case where γ has characteristic 2, and in particular where it has only two elements; here the conic is not self-dual, as all its tangents pass through a fixed point, and every line through this point is a tangent, though its point of contact may not belong to γ but to a quadratic extension of γ. Another peculiar result is that the tangent lines to a general quadric surface (in the field of order 2) form a linear congruence; the surface has a unique tangent in each of its nine points, and these are all the lines meeting a particular pair of skew lines, which are the only lines in space not meeting the quadric at all. Most of the rest of the chapter exhibits how much of the ordinary theory of quadrics in the real field remains valid for an arbitrary field, so long as the latter is not of characteristic 2; as a typical result of this kind we may quote the theorem that every quadratic form in n variables is equivalent in γ (i.e., can be transformed by a reversible linear transformation with coefficients in γ) to one of the form

$$y_1y_1' + \cdots + y_hy_h' + \phi(z_1, \cdots, z_k)$$

where the y's, y''s, and z's are some or all of the new variables (so that $2h+k\leqq n$) and the equation $\phi(z_1, \cdots, z_k)=0$ has no solution in γ except $(0, \cdots, 0)$; moreover, if two expressions in this form are equivalent in γ, the numbers h, k are the same for both (this is the generalisation of Sylvester's theorem) and the forms ϕ are equivalent in γ.

The second chapter deals mainly with applications of geometry to the problem of finding rational solutions of given algebraic equations or systems of equations. As a preliminary example, a set of quadratic and cubic equations in six variables is studied, and shown to define a reducible curve in S_5, whose constituents are examined separately for rational points. Methods applicable to the general cubic in four variables are given, depending largely on the theory of the lines on the cubic surface; these exhibit ten possibilities in an arbitrary field, of which only one (the familiar figure of 27 lines) occurs in an algebraically closed field, and only four in the real field, but all ten in the rational field. Results are quoted by which the equation $\sum_{i=1}^{n} x_i^3=0$ can be solved completely (by polynomials with integer coefficients in $n-2$ parameters) except when $n=3, 5, 7$, or 9. A number of quartic forms in four variables, and the "biaxial" equation $\phi(x, y)=\psi(z, w)$ (where ϕ, ψ are any two forms of the same order, with nonzero discriminants) are treated in something the same manner. Finally some study is given to what the author calls Severi-Brauer varieties, namely those which are birationally equivalent to linear spaces, either in γ or in some extension of γ, the transformation being without exceptional points. The main problems here are, under what circumstances the transformation can be made in γ; if it cannot, what is the least extension of γ in which it can be made; and what is the simplest model (in some defined sense, such as a minimum order model) to which such a variety can be reduced by a birational transformation in γ, without exceptional points. In the study of these problems, the anticanonical system is the tool chiefly used.

This leads naturally up to the third chapter, which is mainly devoted to problems of rationality, especially to the type of extension of γ which may be needed for the rational parametrisation of unirational or birational varieties, defined in γ. Recent results on the general complete intersection of any number of quadrics are given; this variety is birational if the number of variables is sufficiently great compared with the number of equations. The book ends with the problem of under what circumstances an isomorphism between transcendental extensions of two fields implies the existence of, or (more strongly) is necessarily itself an extension of, an isomorphism between the ground fields; and with some applications of the results of Chapter I to the study of the birational self-transformations of a surface by means of the quadratic intersection form of its base.

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