sional slip of this kind.

Again the continual use of the phrase *the necessary and sufficient condition* instead of *a necessary and sufficient condition* in the statement of theorems is rather irritating. After all, the definite article is out of place until after the particular necessary and sufficient condition under consideration has somehow been uniquely characterized, and this is almost never the case prior to the statement of such theorems. The theorems themselves perform this function.

The book is well equipped with clearly designated summaries at the end of most of the sections.

There is a short bibliography at the end. The reviewer regrets that one of his favorite treatises on the subject has been omitted from this bibliography, namely *Lezioni di meccanica razionale* by Levi-Civita and Amaldi.

D. C. LEWIS

The theory of valuations. By O. F. G. Schilling. (Mathematical Surveys, no. 4.) New York, American Mathematical Society, 1950. 8+253 pp. $6.00.

In the words of the author, the theory of valuations may be viewed as a branch of topological algebra. In fact, historically speaking, it represents the first invasion of topology, more precisely, of early metric topology, into the domains of algebra. The introduction of metric methods into algebra has been so fruitful that today many of the deeper algebraic theories carry their mark. In this regard, one should distinguish between the classical use in algebra of the natural metric of the real or complex number fields, such as in proving the "fundamental theorem of algebra," and the much more recent use of the far less evident metrics which are derived from arithmetic notions of divisibility and which constitute the principal notion of valuation theory. Such a metric occurs for the first time in Hensel's construction of the p-adic numbers, dating from the beginning of this century.

The first abstract definition of a valuation was given by Kürschák in 1913. The systematic development of valuation theory is due chiefly to Ostrowski and Krull to the continuation of whose work the author of the present book has made considerable contributions. From about 1920 onwards, valuation theory has played an important part in the theory of algebraic numbers, for instance in the reformulation and completion by Artin, Chevalley, and Hasse of the class field theory, and in the classification of the simple algebras over algebraic number fields by Hasse and Albert. Valuation theory proper and closely related other topological methods have played a fundamental rôle in Krull's work on the structure of commutative rings, and in the building of rigorous foundations for algebraic geometry, notably in the work of Zariski.

It is evident that the ramifications of valuation theory today are so extensive that no single book could possibly do justice to them all. The present book concentrates on the general significance of valuation theory for the algebraic and arithmetic structure of fields, division rings, and simple algebras, including topics such as the local class field theory and non-commutative ideal theory. It does not enter, for instance, into the interplay between absolute values and p-adic values which rules the global aspects of algebraic number theory, and refrains entirely from entering into the applications of valuation theory to general algebraic geometry. In order to indicate the actual scope of the book it will be best to review, briefly and partially, the content of each chapter.

Chapter 1 gives the formal concepts of valuation theory for division rings with unrestricted value groups, and the elementary results concerning the characterization of valuation rings, the relationships between valuations and homomorphisms, and the prolongations of valuations to algebraic extensions.

Chapter 2 is concerned with various notions of completeness of valuated fields, such as ordinary sequential completeness, F. K. Schmidt's extrinsic definition of maximal completeness, and finally a slightly less restrictive notion of completeness which is so designed that the principal tool of valuation theory, Hensel's arithmetic reducibility criterion for polynomials, can be proved inductively. The prolongation theory which follows is based on Ostrowski's notion of relative completeness, that is, on the hypothesis that Hensel's criterion is valid.

Chapter 3 deals with the ramification theory of valuations which is a generalization of Hilbert's ramification theory for algebraic number fields. The problems of this theory concern certain arithmetically defined subgroups of the Galois group of a field extension, the arithmetic and algebraic properties of the corresponding subfields, and their relationships to the value groups and residue class fields. The theory is presented here in a very general form applying to infinite extensions of general relatively complete fields. Free use is made of Krull's Galois theory for infinite extensions. The chapter also contains existence theorems for field extensions with prescribed features such as Galois group, maximal inertial subfield, value group, and residue class field. There are also some results on the structure of the Galois groups of certain special classes of extensions.

Chapter 4 treats the classical ideal theory which governs the arithmetic of algebraic number fields and fields of algebraic functions of one variable. E. Noether's system of axioms for ideal theory is replaced here by a system of postulates concerning the existence of a set of discrete valuations of rank one with certain finiteness and independence properties. The system is so devised that omission of one of the postulates leads to Artin's theory of quasi-divisibility which has been used in algebraic geometry. The chapter culminates in a description of the totality of fields whose arithmetic is that of the classical ideal theory. Besides the fields mentioned above, this includes certain infinite algebraic extensions of them. These results embody earlier work by MacLane and the author.

Chapter 5 contains the generalization to simple algebras of the ideal theory of Chapter 4. The main object is to show that the arithmetic of simple algebras whose centers have the classical ideal theory conforms to Asano's axiomatic pattern for non-commutative ideal theory. Another important topic of this chapter is the analysis of the class group of simple algebras over a relatively complete field. The results are more complicated and less explicit than those for the special case of algebras over p-adic fields, partly because the class group of the simple algebras over the residue class field, which is trivial in the p-adic case, enters the description.

Chapter 6 gives a generalized version of the local class field theory which establishes a correspondence between the abelian extensions of a complete field F with subgroups of the multiplicative group of F. As in the work of Moriya and Nakayama, it is assumed that the base field F is complete with respect to a discrete valuation of rank one, and that the residue class field is perfect and admits for each positive integer n exactly one cyclic extension of degree n. Under these conditions, the division algebras over F can be enumerated in a very simple fashion, and the correspondence of the local class field theory is established by means of the theory of simple algebras as was done by Chevalley for the p-adic fields. The theory is generalized further to cover infinite extensions, and a number of partial generalizations in several directions are given.

Chapter 7 may be regarded as a continuation of Chapter 2. The main concern is with the algebraic and topological structure of complete fields. It contains the Pontrjagin-van Dantzig-Jacobson theory of locally compact and totally disconnected topological division rings, Kaplansky's theory of maximally complete fields which is a generalization of the theory of Hasse, Schmidt, Teichmüller, and Witt for the case in which the valuation is discrete and of rank one, various equivalence problems for complete valuated fields, and a description of the structure of fields which are complete for a valuation of rank one and have a given residue class field.

In addition, there are two auxiliary appendices, one on the Galois theory for infinite extensions, and one on the general theory of linear algebras. Each chapter has a separate bibliography which greatly facilitates the approach to the extensive literature on the various aspects of valuation theory.

Throughout the book, a great effort is made to present the theory with as much generality as is feasible today. Frequently this is accomplished at the expense of much of the intuitive content of the main results. The presentation is extremely concise, unfortunately to the degree of sometimes omitting the easier parts of a proof, and occasionally of suppressing badly needed explanations of terminology and notation. Although these features will cause some difficulty to the general reader, they will not detract from the great value of this book to the specialist for whom it is apparently designed. It fully accomplishes its main purpose, namely to give a systematic account of the present status of valuation theory without dwelling on its applications to other fields. Indeed, a very considerable amount of recent research, a good deal of which is due to the author himself, is here collected in the short space of 250 pages.

G. HOCHSCHILD

Pfaff's problem and its generalizations. By J. A. Schouten and W. van der Kulk. Oxford, Clarendon Press, 1949. 16+542 pp. $12.50.

As originally conceived, Pfaff's problem was to integrate a single equation obtained by equating to zero a linear homogeneous differential expression. Subsequently, the number of equations was increased to a finite arbitrary r and the left members made skew-symmetric forms of arbitrary degrees. Once the dimension of the integral variety sought has been specified, the equations on the differentials are equivalent to equations on the first derivatives of the variables with respect to the parameters on the variety. The most recent generalization, first made in the junior author's thesis written under the senior author's direction in 1945, replaces the skew-symmetric equations defining the derivatives implicitly by parametric equations defining them explicitly, subject to certain conditions on the rank of matrices in the first and second derivatives.

This book gives in one of the notations associated with the tensor calculus a unified account of the main results in the area just outlined. The classical theories of first order linear partial differential systems,