every polynomial ideal its "algebraische Mannigfaltigkeit" (AM), a term used here in a sense different from the customary. The AM of an ideal is to consist, namely, of its NG together with certain "infinitely near" loci. But the author does not go beyond a few suggestive remarks, and the main work on the idea remains to be carried out, if it can be at all.

The chapter on syzygy theory is an original contribution of Dr. Gröbner, and has appeared in the Monatshefte für Mathematik vol. 53 (1949) pp. 1–16. A part of this paper has been criticized by P. Dubreil in the Comptes Rendus, Académie des Sciences, Paris vol. 229 (1949) pp. 11–12, and the criticism applies also to some extent to the book. (Dubreil's and Gröbner's papers are reviewed in Mathematical Reviews vol. 11 (1950) p. 489.)

While illustrative examples are occasionally given in the footnotes, Dr. Gröbner could increase the usefulness of his textbook by including exercises and also further references to the literature.

A. SEIDENBERG

The variational principles of mechanics. By Cornelius Lanczos. (Mathematical Expositions, no. 4.) Toronto, University of Toronto Press, 1949. 25+307 pp. $5.75.

The most outstanding feature of this book is the enthusiastic style in which it is written. The enthusiasm is contagious to the extent that even the most iconoclastic reader can not but be intrigued by "Lagrange's ingenious idea" (on p. 39), by d'Alembert's stroke of genius (p. 88), by Gauss's "ingenious reinterpretation" thereof (p. 106), by the "amazing result" of Hamilton (p. 220), or by the biblical quotation at the head of the eighth Chapter: "Put off thy shoes from off thy feet, for the place whereon thou standest is holy ground."

The author gives on the whole an able exposition of the following topics: The Euler-Lagrange equations of the calculus of variations, d'Alembert's principle, the principle of least constraint, the Lagrangian equations of motion, principle of Hamilton, principle of least action, integration by ignoration of coordinates, the Legendre's transformation, the canonical equations of motion, integral invariants, canonical transformations, the brackets of Lagrange and Poisson, infinitesimal canonical transformations, the partial differential equation of Hamilton and Jacobi, solution by separation of variables, Delaunay's treatment of separable periodic systems, the significance of all this material in the development of both the older and more recent quantum mechanics.

In addition to the final chapter devoted to a brief historical survey, the entire treatment is more or less historical. To give an adequate idea of the general spirit and purpose of this mode of presentation, the reviewer can hardly do better than to quote from the preface:

"There is a tremendous treasure of philosophical meaning behind the great theories of Euler and Lagrange, and of Hamilton and Jacobi, which is completely smothered in a purely formalistic treatment, although it cannot fail to be a source of the greatest intellectual enjoyment, to every mathematically-minded person. To give the student a chance to discover for himself the hidden beauty of these theories was one of the foremost intentions of the author. For this purpose he had to lead the reader through the entire historical development, starting from the very beginning, and felt compelled to include problems which familiarize the student with the new concepts. These problems, of a simple character, were chosen in order to exhibit the general principles involved."

In following this historical approach, the author gives, for instance, Euler's original intuitive derivation of the differential equations of the calculus of variations as well as the more usual rigorous treatment of Lagrange. For having done this in an attractive and readable style, the reader will perhaps forgive the author for having fumbled slightly in his treatment of the Lagrange derivation. For in his discussion of the fundamental lemma of the calculus of variations he comes to an approximate equation (p. 59),

$$\frac{\delta I}{\epsilon} = E(\xi)\int_{\xi-\rho}^{\xi+\rho}\phi(x)dx \qquad \left(E(x) = \frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right)$$

with the remark that the error tends to zero as ρ tends to zero. But he fails to remark that the whole integral on the right also apparently tends to zero, and nothing is said to show why the error might not be numerically equal to the whole right-hand side of the equation if $E(\xi)\neq 0$. The author fails even to state explicitly the essential hypothesis that $E(x)$ be continuous, merely stating rather vaguely that it is "practically constant" over the range of integration when ρ is small. Of course, these difficulties can be easily remedied without sacrifice of brevity and, for this reason, it is the greater pity that the remedies were not included.

It is possible to criticize other details of the book. Why, for instance, does the author drag in a half baked discussion of variation on page 38, where the subject under discussion is merely the problem of minimizing a function $f(u)$ of one or more variables? Here, of all places, whatever distinction can, in general, be made between a differential, du, and a "variation," δu, completely disappears; for u is merely an independent variable. Further on, in connection with the minimization of the definite integral $\int_a^b F(x, y, y')dx$, the author again (and this time quite properly) emphasizes on page 55 the distinction between differential and variation. But it seems to the reviewer that he should also emphasize their similarity. In fact, the variation δy *can* be regarded as a differential just as truly as the quantity denoted by dy. For in forming the variation of a function $y(x)$, the latter might be imagined as imbedded in an arbitrary sufficiently smooth family of functions $y(x, \alpha)$, where α is the parameter of the family and where $y(x, 0) = y(x)$. Then the "differential" $dy = y_x(x, 0)dx$, while the "variation" $\delta y = y_\alpha(x, 0)d\alpha$, the subscripts denoting partial differentiations in the usual manner.

Incidentally the wholesale use of infinitesimals, such as occurs in this book, while somewhat traditional, is rather repulsive to the reviewer. Many, if not all, of the theorems and proofs can be formulated in terms of derivatives just as easily as in terms of infinitesimals, differentials, virtual displacements, and variations. Moreover this can probably be done with no loss of brevity and with considerable gain in clarity and rigor. Other things being equal, relations involving derivatives would seem to be preferable to those involving infinitesimals, because the former are simple equalities whereas the latter are encumbered, either explicitly or implicitly, with higher order infinitesimals which we are supposed to neglect. The spectre of an incorrect appraisal of these orders is ever present.

By failing to give such a treatment using derivatives, the author has missed a golden opportunity of filling a real gap in the literature. Perhaps there are still other and more golden opportunities (both missed and taken) which a more astute reader might have detected. But at any rate, much of the author's scintillating style seems to a certain extent wasted on just another text book, whose claims for originality, even in exposition, are strictly limited.

A number of other minor comments may be made:

The author sometimes is so carried away by his own enthusiasm that, seemingly for rhetorical effect, he transgresses the limits of accuracy. At the bottom of page 113, for example, Hamilton's principle is stated as if it applied to an arbitrary mechanical system and the statement is even italicized. That the author knows better than this is indicated in other passages, one of which occurs on the middle of the next page. Of course, the book, though it mentions non-holonomic systems and other so-called polygenic systems, for which Hamilton's principle does not apply, is not primarily concerned with such systems; and so it is possible to excuse the author for an occasional slip of this kind.

Again the continual use of the phrase *the necessary and sufficient condition* instead of *a necessary and sufficient condition* in the statement of theorems is rather irritating. After all, the definite article is out of place until after the particular necessary and sufficient condition under consideration has somehow been uniquely characterized, and this is almost never the case prior to the statement of such theorems. The theorems themselves perform this function.

The book is well equipped with clearly designated summaries at the end of most of the sections.

There is a short bibliography at the end. The reviewer regrets that one of his favorite treatises on the subject has been omitted from this bibliography, namely *Lezioni di meccanica razionale* by Levi-Civita and Amaldi.

D. C. LEWIS

The theory of valuations. By O. F. G. Schilling. (Mathematical Surveys, no. 4.) New York, American Mathematical Society, 1950. 8+253 pp. $6.00.

In the words of the author, the theory of valuations may be viewed as a branch of topological algebra. In fact, historically speaking, it represents the first invasion of topology, more precisely, of early metric topology, into the domains of algebra. The introduction of metric methods into algebra has been so fruitful that today many of the deeper algebraic theories carry their mark. In this regard, one should distinguish between the classical use in algebra of the natural metric of the real or complex number fields, such as in proving the "fundamental theorem of algebra," and the much more recent use of the far less evident metrics which are derived from arithmetic notions of divisibility and which constitute the principal notion of valuation theory. Such a metric occurs for the first time in Hensel's construction of the p-adic numbers, dating from the beginning of this century.

The first abstract definition of a valuation was given by Kürschák in 1913. The systematic development of valuation theory is due chiefly to Ostrowski and Krull to the continuation of whose work the author of the present book has made considerable contributions. From about 1920 onwards, valuation theory has played an important part in the theory of algebraic numbers, for instance in the reformulation and completion by Artin, Chevalley, and Hasse of the class field theory, and in the classification of the simple algebras over algebraic number fields by Hasse and Albert. Valuation theory proper and closely related other topological methods have played a funda-