THE NOVEMBER MEETING IN LOS ANGELES

The four hundred sixty-third meeting of the American Mathematical Society was held at the University of California, Los Angeles, on Saturday, November 25, 1950. The attendance was approximately 100, including the following 76 members of the Society:

T. M. Apostol, Richard Arens, L. A. Aroian, W. G. Bade, E. F. Beckenbach, Clifford Bell, Gertrude Blanch, H. F. Bohnenblust, J. V. Breakwell, F. H. Brownell, A. S. Cahn, Luther Cheo, T. M. Cherry, L. M. Coffin, J. G. van der Corput, A. C. Davis, E. A. Davis, C. R. DePrima, R. J. Dickson, R. P. Dilworth, H. A. Dye, Arthur Erdélyi, Harley Flanders, G. C. Forsythe, J. W. Green, W. T. Guy, G. J. Haltiner, A. R. Harvey, Emma Henderson, J. G. Herriot, M. R. Hestenes, P. G. Hoel, Alfred Horn, R. E. Horton, D. G. Humm, Rufus Isaacs, J. R. Jackson, Fritz John, P. B. Johnson, P. J. Kelly, Cornelius Lanczos, G. E. Latta, R. B. Leipnik, G. F. McEwen, J. C. C. McKinsey, A. V. Martin, A. B. Mewborn, W. E. Milne, T. S. Motzkin, Fritz Oberhettinger, E. E. Osborne, R. H. Owens, L. J. Paige, George Pólya, W. T. Puckett, H. R. Pyle, W. P. Reid, Edgar Reich, R. R. Reynolds, Seymour Sherman, G. E. F. Sherwood, E. S. Sokolnikoff, R. H. Sorgenfrey, M. L. Stein, Robert Steinberg, E. G. Straus, A. C. Sugar, J. D. Swift, A. E. Taylor, F. G. Tricomi, F. A. Valentine, D. D. Wall, S. S. Walters, L. F. Walton, Morgan Ward, W. R. Wasow, G. M. Wing, F. H. Young.

In the morning there was a general session for contributed papers at which Professor E. F. Beckenbach presided. At the conclusion of this session, brief reports were made by Professor E. F. Beckenbach and Professor František Wolf on the progress of the Pacific Journal of Mathematics. In the afternoon Professor Erdélyi introduced Professor van der Corput, who presented the invited address, *Asymptotic expansions*. This was followed by two sessions for contributed papers, at which Professors A. E. Taylor and L. F. Walton presided.

Following the meeting there was a tea at the Institute for Numerical Analysis of the National Bureau of Standards. On display for the visitors were the extensive computing machinery of the Institute and the offices of the Pacific Journal.

Abstracts of papers presented at the meeting follow. Papers presented by title are indicated by the letter "*t*." Paper number 56 was read by Professor Apostol.

ALGEBRA AND THEORY OF NUMBERS

56. T. M. Apostol and H. S. Zuckerman: *On magic squares constructed by the uniform step method.*

If the numbers $1, 2, \cdots, n^2$ are placed in a square array such that any row or column contains a complete residue system mod n, but no two numbers between the same two multiples of n, then the square is called regular. Every regular square is magic. Let (A, B) be the coordinates of the square in the Ath column and Bth row. To construct the square, 1 is placed in cell (p, q). If x is in (A_x, B_x), then in general $x+1$ is placed in $(A_x+\alpha, B_x+\beta)$ by making the "step" (α, β) from (A_x, B_x). If this cell is already occupied, then a "break-step" (a, b) is used to put $x+1$ in $(A_x+\alpha+a, B_x+\beta+b)$ instead. If this cell is also occupied then a second break-step (c, d) is used. Let $D(\alpha, n)=D(\beta, n)=\delta$, $D(a, n)=D(b, n)=\epsilon$, $D(\alpha b-a\beta, n)=\delta\epsilon\theta$, $D(\delta, \epsilon)=1$, where D denotes the g.c.d. Then the method above is described by the congruences $A_x\equiv p+\alpha(x-1)+a[(x-1)\delta/n]+c[(x-1)\delta\epsilon\theta/n^2]$, $B_x\equiv q+\beta(x-1)+b[(x-1)\delta/n]+d[(x-1)\delta\epsilon\theta/n^2]$ (mod n). From these congruences the authors prove: The square will be filled if and only if $D(bc-ad, \delta\epsilon\theta)=\epsilon$, $D(\beta c-\alpha d, \delta\epsilon\theta)=\delta$. If $\theta=1$, the resulting square will be regular if and only if $D(c, \delta\epsilon)=D(d, \delta\epsilon)=1$. The case $\delta=\epsilon=\theta=1$ was considered by D. N. Lehmer (Trans. Amer. Math. Soc. vol. 31 (1929) pp. 529–551). (Received October 11, 1950.)

57. Anne C. Davis: *A fixpoint criterion for completeness of a lattice.* Preliminary report.

Let L be a lattice with the inclusion relation $\leqq$. As is known (Birkhoff, *Lattice Theory*, rev. ed., p. 54; Tarski, Bull. Amer. Math. Soc. Abstract 55-11-496), for L to be complete it is necessary that every increasing function on L to L have a fixpoint. Tarski asked the question whether this condition is also sufficient. The answer is affirmative. In fact, if L is incomplete, there are sets $X\subseteq L$ with no least upper bound; let M be such a set with the smallest possible power. Using the well-ordering principle, two transfinite sequences, α and β, are constructed such that: (i) α is increasing, every term of α is the least upper bound of some set $X\subseteq M$, and every upper bound of α is an upper bound of M; (ii) β is decreasing, every term of β is an upper bound of M, and no lower bound of β is an upper bound of M. Now let $x\in L$. If x is an upper bound of M, let $f(x)$ be the first term β_η of β with x not $\leqq \beta_\eta$; otherwise, let $f(x)$ be the first term α_ξ of α with α_ξ not $\leqq x$. The function f thus defined is increasing and has no fixpoint. (Received October 12, 1950.)

58t. D. G. Duncan: *A problem in modular lattices. (Problem 29, G. Birkhoff, Lattice theory, rev. ed.)*

Denote by C the lattice obtained by embedding the lattice $1+1=FL(2)$ in a finite chain. The free modular lattice generated by $2+C$ is determined by constructing the meet and join tables of its elements. The free modular lattice generated by $2+1+1$ (Problem 29) then appears as a special case of this more general problem. (Received October 12, 1950.)

59. Harley Flanders: *Some matrix theorems.*

First result: If A is a normal matrix, then A^* is expressible as a polynomial in A. More generally, if $A_1, \cdots, A_m$ are normal transformations on a finite-dimensional space, then there is a complex polynomial $f(x)$ such that $A_i^*=f(A)$. Second result: Let A be an m by n and B an n by m matrix with coefficients in an abstract field k. Then the nonzero characteristic roots of AB and BA coincide with the same *geometric* multiplicities. What is more, the corresponding elementary divisors are the same. (Received November 25, 1950.)

60. L. J. Paige: *Complete mappings of finite groups.*

A biunique mapping $x \to \theta(x)$ of a group G upon G is said to be a complete mapping if $x \cdot \theta(x) \equiv \eta(x)$ is a biunique mapping of G upon G. A necessary condition that there exist a complete mapping for a group G is that there exist an ordering of the elements of G such that the product of the elements in this order is the identity. If H is an invariant subgroup of G such that H and G/H have complete mappings, then G has a complete mapping. (Received October 10, 1950.)

61t. Alfred Tarski: *On homomorphic images of complete Boolean algebras.* Preliminary report.

Let $\mathfrak{P}(X)$ denote the power of the set X. Given a cardinal $\aleph$, a Boolean algebra B is *$\aleph$-complete* if the join $\sum X$ exists for every $X \subseteq B$ with $\mathfrak{P}(X) \leqq \aleph$; an ideal I in B is *$\aleph$-complete* if $\sum X \in I$ for every $X \subseteq I$, $\mathfrak{P}(X) \leqq \aleph$. Let B be an $\aleph_0$-complete Boolean algebra, I any ideal in B, and B/I the corresponding coset algebra. Then the following results hold: I. *If $X \subseteq B/I$ is a set of disjoint elements, $\mathfrak{P}(X) \leqq \aleph_0$, and $\sum X$ exists (in B/I), then $\sum Y$ exists for every $Y \subseteq X$.* II. *If $\mathfrak{P}(X) \leqq \aleph_0$ for every set $X \subseteq B/I$ of disjoint elements, then B/I is complete.* III. *If m is any measure on B, and I the set of all $x \in B$ with $m(X) = 0$, then B/I is complete.* IV. $\mathfrak{P}(B/I) \neq \aleph_0$. II and III were previously obtained under additional assumptions: I is $\aleph_0$-complete, m is countably additive (for references see Tarski, Comtes Rendus des Séances de la Société des Sciences et des Lettres de Varsovie vol. 30, Class III, pp. 175 ff.). IV implies that no countably infinite Boolean algebra is a homomorphic image of a complete atomistic Boolean algebra (a solution of a problem of Dilworth). Assuming that B is $\aleph_{\xi+1}$-complete and I is $\aleph_\xi$-complete (ξ an arbitrary ordinal), I and II remain valid if $\aleph_0$ is replaced by $\aleph_{\xi+1}$. (Received October 13, 1950.)

62. Morgan Ward: *A class of soluble diophantine equations.*

The diophantine equation (D): $Z^m = F(x)$ is considered over a commutative ring R with unit element. Here $F(x)$ is a homogeneous form in k indeterminates $x_1, x_2, \cdots, x_k$ with coefficients in R of degree n, m is a positive integer, and z is another indeterminate. In case m is prime to n, there exists in R a k-parameter family of solutions of (D) which may be explicitly exhibited given m, r, and $F(x)$. In case R is a field, this family essentially includes all solutions of (D) in R with $z \neq 0$. For example, the diophantine equation $x^n + y^n = z^m$ has a two-parameter family of integral solutions for every m prime to n. (Received November 6, 1950.)

Analysis

63. W. G. Bade: *The inversion of generalized convolution transforms.*

Let $E(s) = e^{bs} \prod_1^\infty (1 - s/a_k) e^{s/a_k}$, a_k real, $\sum_1^\infty 1/a_k^2 < \infty$, and let $G(t)$ be the inverse bilateral Laplace transform of $1/E(s)$. Widder and Hirschman (Trans. Amer. Math. Soc. vol. 66 (1949) pp. 135–201) have shown that in the space $C[-\infty, \infty]$ (and more generally) the operator $\int_{-\infty}^\infty G(\xi) x(t-\xi) d\xi$ is inverted by $\lim_{n\to\infty} e^{bD} \prod_1^n (1 - D/a_k) e^{D/a_k}$ where $D = d/dt$, $e^{aD} x(t) = x(t+a)$. Essential use is made of the fact $G(t) \geqq 0$, and various methods are needed for different distributions of the $\{a_k\}$. Their results for $C[-\infty, \infty]$ are covered by the following theorem, which does not require that $G(\xi)$ be real. Theorem: Let X be a complex Banach space. Let T be a closed distributive (generally unbounded) operator satisfying (a) the domain $D(T)$ is dense in X, (b) $\sigma(T)$ is confined to the strip $-\alpha < \sigma < \alpha < \infty$, $s = \sigma + i\tau$, (c) the resolvent $(sI - T)^{-1} = R_s(T)$

satisfies $\|R_s(T)\| \leqq 1/|\sigma-\alpha|$. Let $E(s)$, $P_n(s)$ $(P_0\equiv 1)$ satisfy (1) $P_n(s)$ are polynomials in s and e^{as}, (2) $P_n(s)\rightarrow E(s)$ and $|P_n(s)/E(s)| \leqq M$ for s in $-r<\sigma<r$, $r>\alpha$, (3) $P_n(s)/E(s)$ are absolutely convergent bilateral Laplace transforms in $-r<\sigma<r$ of functions $G_n(\xi)$, $-\infty<\xi<\infty$, $(G_0=G)$, (4) the transformations defined by $\int_{-\infty}^{\infty} G_n(\xi)e^{-\xi T}xd\xi$, $x\in X$, have uniformly bounded norms. Then if $y=\int_{-\infty}^{\infty} G(\xi)e^{-\xi T}xd\xi$, $x\in X$, $\lim_{n\to\infty} P_n(T)y=x$. For definition of $e^{\xi T}$ see Hille, Amer. Math. Soc. Colloquium Publications, vol. 31, chap. 12. Convergence is proved first for $x\in D(T^2)$ and extended using the Banach-Steinhaus Theorem. (Received November 7, 1950.)

64. F. H. Brownell: *Asymptotically ergodic output under ergodic input of delay differential machines.*

Consider here the behavior of a linear autonomous delay differential machine subjected to an ergodic input, the governing equation thus being (1) $x^{(n)}(t) + \sum_{k=0}^{n-1} \int_0^{\infty} x^{(k)}(t-h)dF_k(h) = y(t)$ where $y(t)$ is the input and $x(t)$ the output. If the machine coefficient functions $F_k(h)$ satisfy a few reasonable conditions, and if the resulting system is stable, one can show that if $y(\xi, t)$ is an ergodic process for ξ over a probability space, then $x(\xi, t) = x_1(\xi, t) + x_2(\xi, t)$ where $x_1(\xi, t)$ decays exponentially for each ξ and $x_2(\xi, t)$ is an ergodic process. Also the power spectrum of $x_2(\xi, t)$ can be obtained as one would expect formally from that of $y(\xi, t)$. The proof uses the Laplace transform representation of the solution of (1), which can easily be justified, and from which the conclusion is almost trivial. The result is implicit for the significance of much of Wiener's "time series." Doob in the Berkeley Symposium, 1945–1946, proves for equation (1) without delay terms that a stationary input implies an asymptotically stationary output, but does not discuss ergodicity. (Received October 23, 1950.)

65. H. A. Dye: *A Radon-Nikodým theorem for operator algebras.*

Let $\mathfrak{A}$ be a W^*-algebra of operators on a complex Hilbert space, $\mathfrak{A}$ being of finite class and satisfying the condition that any family of mutually orthogonal nonzero projections is at most countably infinite. A positive functional ρ on $\mathfrak{A}$ is called countably additive in case $\rho(\sum P_n) = \sum \rho(P_n)$, for any sequence of mutually orthogonal projections in $\mathfrak{A}$. An absolute continuity relation $<$ is introduced between countably additive positive functionals by defining $\rho<\sigma$ in case $\sigma(P)=0$ implies $\rho(P)=0$, for any projection P in $\mathfrak{A}$. Under these conditions the following Radon-Nikodým theorem holds: $\rho<\sigma$ if and only if $\rho(A)=\sigma(T^*AT)$, for some closed densely-defined operator $T(\eta\mathfrak{A})$ and all A in $\mathfrak{A}$. A justification of this formal equation is contained in the key result that any such functional has the form $\sum_{i=1}^{\infty} (Ax_i, x_i)$, for appropriate vectors x_i. As corollaries, one proves the following: (1) the trace T on a finite ring satisfies an equation $\sum_{i=1}^{\infty} (T(A)x_i, x_i) = \sum_{i=1}^{\infty} (Ax_i, x_i)$; (2) the strongest and the strong neighborhood topologies coincide on any finite ring; (3) any algebraic *-isomorphism between finite rings is bicontinuous in the weak neighborhood topology. (Received November 6, 1950.)

66. Alfred Horn: *Symmetrisable operators.*

A bounded linear operator T in Hilbert space is said to be symmetrisable if there exists a positive symmetric bounded operator G such that GT is symmetric. It is proved that such an operator T always has values different from 0 in its spectrum provided that $GT\neq 0$. The proof carries over to the case of *-algebras. In case T is completely continuous, and $Gx=0$ implies $Tx=0$, A. C. Zaanen has given an expansion for Tx in terms of the eigen-vectors of T. Another proof is obtained here. (Received October 12, 1950.)

67. Rufus Isaacs: *The convergence of monodiffric power series.*

Monodiffric functions of a complex variable are those which satisfy partial difference equations analogous to the Cauchy-Riemann equations and thus constitute a finite difference analogue of classical analytic functions. They include a class of series corresponding to power series which can also be viewed as a complex extension of the Newton interpolatory series. Their region of convergence is as follows: A certain point $z_0 = x_0 + iy_0$ in the strip $|x-y| \leqq 1$ is called the vertex of convergence. The series converges in the quarter-plane $x > x_0$, $y > y_0$; also (if they are not already included) the half-lines x = nonnegative integer, $y > y_0$ and y = n.n.i., $x > x_0$, and the points where both x and y = n.n.i.'s. The series diverges in the interior of the remaining region with two possible exceptions. There is a segment extending downwards and leftwards from z_0 at 45° of length always less than 3/2 which is not covered by the present proof; certain special series where only every fourth coefficient is not zero may converge at every fourth integral lattice point on the line $x = y$. (Received November 22, 1950.)

68. R. B. Leipnik: *Size functions.*

Let S be a set, and for each cardinal a, let $S(a)$ be the set of a-membered subsets of S. A function f is called an a-size function in S if and only if f is on S (successor a) to the non-negative finite reals, $\sum_{x \in T} f((T - \{x\}) \cup \{y\}) \geqq f(T)$ whenever $y \in S$ and $T \in S$ (successor a), and $f(U) > 0$ if and only if cardinal (U) = cardinal (distinct (U)), where distinct (U) is the set of distinct elements of U. (A 1-size function is a metric.) The relation of size functions to measure and dimension theory is developed by metric techniques. (Received October 12, 1950.)

69. J. J. Newman: *Remark on some papers by H. Pollard.*

This note gives a short proof that there exists a function $f(x)$ of class L^4 such that the Legendre series of $f(x)$ fails to converge in L^4 mean. This settles a question raised by H. Pollard, who showed (Trans. Amer. Math. Soc. vol. 62 (1947) pp. 387–403) that the same property obtains for L^p if $p > 4$ but not if $2 \leqq p < 4$. The analogous questions for ultraspherical and Jacobi series (Pollard, loc. cit. vol. 63 (1948) pp. 355–367, and Duke Math. J. vol. 16 (1950) pp. 189–191) have a similar answer. (Received October 16, 1950.)

70. F. G. Tricomi: *Expansion of the hypergeometric functions in series of confluent ones.*

Putting $x = 2z/(c-2a+z)$ Gauss hypergeometric function $F(a, b; c; x)$ may be developed in an infinite series of confluent hypergeometric functions, $F_1(c-b, c+m; z)$, $m = 0, 1, 2, \cdots$, whose coefficients are connected in a very simple manner with the coefficients A_m of this author's known expansion of the confluent functions in series of Bessel functions. The series converges inside the circle $|z| < |c-2a|$. Applied to the particular case of the Jacoby polynomials the new expansions furnish very good and simple asymptotic approximations of the *first* and *last* zeros of these polynomials, which even in the case of the Legendre polynomials are better than those given in the past. (Received October 4, 1950.)

71. F. H. Young: *Matrix transformations of Fourier coefficients.*

Let $T=(a_{kj})$ be an $\infty\times\infty$ matrix of real constants, where $\sum_{j=0}^{\infty}|a_{kj}|<M$ for all k. Let $f\sim(a_k, b_k)$, f in L_p for some p, $1\leqq p\leqq\infty$. T is said to belong to the class (L_p) if $(\sum_{j=0}^{\infty}a_{kj}a_j, \sum_{j=0}^{\infty}a_{kj}b_j)\sim F$ in L_p. Necessary and sufficient conditions are developed for T to belong to (L_p), with special emphasis on the end points (L_1) and (L_∞). Certain special results are obtained from the use of symmetric matrices. Matrix transformations are generalizations of those effected by factor sequences, which correspond to diagonal matrices. (Received October 2, 1950.)

Geometry

72. F. A. Valentine: *A characterization of simply-connected closed arcwise convex sets.*

Let S be a set of points in the Euclidean plane E_2. A set $S\subset E_2$ is said to be *unilaterally connected* if for each pair of distinct points x and y in S, there exists a continuum $S_1\subset S$ which contains x and y, and which lies in one of the closed half-planes determined by the line $L(x, y)$. (The line $L(x, y)$ is determined by x and y.) A set $S\subset E_2$ is said to be *arcwise convex* if each pair of points in S can be joined by a convex arc lying in S. (A convex arc is one which is contained in the boundary of its convex hull.) *Theorem: A necessary and sufficient condition that a simply-connected closed set $S\subset E_2$ be arcwise convex is that it be unilaterally connected.* (Received October 11, 1950.)

Logic and Foundations

73t. Alfred Tarski: *On a statement related to the principle of choice.* Preliminary report.

Let A be any set, B the set of all its nonempty subsets, and C the set of all functions on B to A. The fact that the principle of choice does not apply to subsets of A can be expressed as follows: I. *For every $f\in C$ there is an $X\in B$ such that $f(X)\notin X$.* Consider the statement: II. *There is a function G on C to B such that $f(G(f))\notin G(f)$ for every $f\in C$.* II has the form of a special case of the principle of choice. Clearly II implies I. Henkin, who formulated II, raised the problem whether II can be derived from I (without the help of the principle of choice). The solution is affirmative. In fact, given an $f\in C$, we define by transfinite induction: $x_0=f(A)$, $x_\beta=f(A-\{x_\xi\}_{\xi<\beta})$ unless $A-\{x_\xi\}_{\xi<\beta}$ is empty. By I we conclude that there is smallest α such that $A-\{x_\xi\}_{\xi<\alpha}$ is not empty and $x_\alpha\notin A-\{x_\xi\}_{\xi<\alpha}$. We let $G(f)=A-\{x_\xi\}_{\xi<\alpha}$. (Transfinite ordinals can be eliminated from this argument by the method used by Zermelo in the proof of the well ordering principle.) Thus, II *is equivalent to* I, and *the statement that* II *holds for no set A is equivalent to the axiom of choice.* (Received October 13, 1950.)

74t. Alfred Tarski: *Remarks on the formalization of the predicate calculus.* Preliminary report.

The lower predicate calculus with identity can be formalized as indicated in Quine's *Mathematical logic*, chap. 2, by adding: *I. $\vdash\ulcorner\alpha=\alpha\urcorner$, and *II. $\vdash\ulcorner\alpha=\beta\supset(\Phi\supset\Psi)\urcorner$ *where Φ is atomic and Ψ is obtained from Φ by replacing an occurrence of α by β*. This formalization can be equivalently simplified by replacing (Quine's) *103 by *103′: $\vdash\ulcorner\Phi\supset(\alpha)\Phi\urcorner$ *for α not occurring in Φ*; *104 by *104′: $\vdash\ulcorner(\alpha)\Phi\supset\Phi\urcorner$; *I by *I′: $\vdash\ulcorner\sim(\alpha)\sim(\alpha=\beta)\urcorner$. In this new version the notion of proper substitution of free variables is not involved. The notion of free variables can be dispensed with entirely by agreeing to consider as theorems all formulas whose closures are theorems in the old sense; to formalize the predicate calculus thus modified, omit all mention of closure in *100, *101, *102, *103′, *104′, *I′, *II, and accept (besides modus ponens, *105) a new rule of inference: *If ϕ is a theorem, so is* $\ulcorner(\alpha)\Phi\urcorner$. Intuitively underlying the preceding remarks is the following simple fact: *If Ψ is like Φ except for containing free occurrences of β whenever Φ contains free occurrences of α, then* $\vdash \ulcorner\Psi \equiv (\alpha)(\alpha=\beta \supset \Phi)\urcorner$, that is, $\ulcorner(\alpha)(\alpha=\beta\supset\Phi)\urcorner$ is an equivalent expression for Ψ. The above observations are relevant for the foundations of many-dimensional projective algebras. Compare Abstract 57-1-75 by Thompson. (Received October 23, 1950.)

75t. F. B. Thompson: *Some problems concerning the predicate calculus.* Preliminary report.

K is the class of theorems of the predicate calculus which remain theorems when atomic formulae are uniformly replaced by atomic formulae with the same variables. Let $L\subset K$ contain those theorems which remain theorems when atomic formulae are uniformly replaced by arbitrary atomic formulae. Among axioms for predicate calculus in Quine's *Mathematical logic*, *100–*102 are in L, *103 is in K but not in L, *104 is in neither. Let *103′ be $\vdash \ulcorner(\alpha)\Phi\supset(\alpha)(\alpha)\Phi\urcorner$ and $\vdash \ulcorner\sim(\alpha)\Phi\supset(\alpha)\sim(\alpha)\Phi\urcorner$, and *104′ be $\vdash \ulcorner(\alpha)\Phi\supset\Phi\urcorner$. *103′ and *104′ are in L. Tarski asked the questions: Are all theorems (i) in K, (ii) in L, derivable from (i) *100–*103, *104′, (ii) *100–*102, *103′, *104′? The theorems which are closures of $\ulcorner(\alpha)\Phi\wedge(\beta)\Theta\wedge(\gamma)\Psi\supset(\alpha)(\beta)(\gamma)\cdot\{(\alpha)[(\beta)\Psi\wedge(\gamma)\Theta]\wedge(\beta)[(\gamma)\Phi\wedge(\alpha)\Psi]\wedge(\gamma)[(\alpha)\Theta\wedge(\beta)\Phi]\}\urcorner$, α, β, γ being free in Φ, Θ, and Ψ, are counter-examples for both cases. A problem formulated by Jaśkowski, Studia Societatis Scientiarum Torunensis. Sect. A vol. 1, p. 19 (see Henkin, Journal of Symbolic Logic vol. 14, p. 65), is equivalent to problem (ii), and thus has a negative solution. In algebraic translation, the above result gives a negative solution of the representation problem for 3-dimensional projective algebras. (Many-dimensional projective algebras were defined by Chin-Tarski. For 2 dimensions see Everett-Ulam, Amer. J. Math. vol. 68, and Chin-Tarski, Bull. Amer. Math. Soc. Abstract 54-1-90.) (Received October 13, 1950.)

Statistics and Probability

76. Edgar Reich: *On the definition of information rate.*

Consider a two-point unidirectional communication system where $p(x, y)$ is the probability that a random variable x is transmitted, and as a result of this, a random variable y, not necessarily the same as x if there is "noise," is received. We have $p_0(x)=\int p(x, y)dy$ as the a priori probability that x *will* be transmitted, and $p_y(x)=p(x, y)/\int p(x, y)dx$ as the a posteriori probability that x *was* transmitted. Postulate the existence of $F(u, v)$ such that $U\{p\}=\int F[p(x), x]dx$ is a measure of the uncertainty associated with the distribution $p(x)$. Then $I=U\{p_0(x)\}-U\{p_y(x)\}$ is a measure of the information gained on receiving y, and $R=E[I]$ is the rate at which information is received. Require the invariance of R under any one-to-one relabeling of the variables $\xi=f(x)$, $\eta=f(y)$. Under mild restrictions, it follows that $R=\int\int p(x, y)\cdot\log [p(x, y)/p_0(z)q(y)]dxdy$, where $q(y)=\int p(x, y)dx$. This is the formula proposed by C. E. Shannon from other considerations. (Received September 18, 1950.)

Topology

77t. S. T. Hu: *Reduction of the homotopy sequences of the spaces of*

paths to the exact sequences concerning the triad homotopy groups.

A path σ in a topological space X is a map $\sigma: I \to X$ of the unit interval I into X. Let Ω denote the space of all paths in X with the compact-open topology. Let A, B be subsets of X with nonvacuous intersection $C = A \cap B$ and $x_0 \in C$. Denote by $[X, A, B]$ the subspace of Ω consisting of the paths σ with $\sigma(0) \in A$ and $\sigma(1) \in B$ and analogously for analogous notations. The point path $\sigma_0(I) = x_0$ is used as the basic point for all homotopy groups concerned. First, the homotopy sequence of the pair $[X, x_0, B]$ and $[A, x_0, C]$ reduces to the exact sequence $\cdots \to \pi_n(A, C) \to \pi_n(X, B) \to \pi_n(X; A, B) \to \pi_{n-1}(A, C) \to \cdots$ of Blakers and Massey. Next, let Γ denote the subset of $[X, x_0, x_0]$ consisting of the paths σ with $\sigma(t) \in A$ if $t \leqq 1/2$ and $\sigma(t) \in B$ if $t \geqq 1/2$. The homotopy sequence of the pair $[X, x_0, x_0]$ and Γ reduces to the exact sequence $\cdots \to \pi_n(A/B) \to \pi_n(X) \to \pi_n(X; A, B) \to \pi_{n-1}(A/B) \to \cdots$ of Blakers and Massey (Bull. Amer. Math. Soc. Abstract 56-2-208). Last, assume that A and B be contractible. Then the homotopy sequence of the pair $[X, A, B]$ and $[C, C, C]$ reduces to a new exact sequence $\cdots \to \pi_{n+1}(X; A, B) \to \pi_{n-1}(C) \to^{s} \pi_n(X) \to \pi_n(X; A, B) \to \cdots$. The homomorphism $s: \pi_{n-1}(C) \to \pi_n(X)$ is a generalisation of Freudenthal's suspension operation. (Received October 11, 1950.)

78*t*. S. T. Hu: *A fibering of the space of paths over the space of curves in a metric space.*

Consider a metric space X with metric ρ and two subsets A and B. Let Ω denote the totality of the paths (equal to p-curves) $f: I \to X$ in X with $f(0) \in A$ and $f(1) \in B$. Ω forms a metric space with metric ρ defined by $\rho(f, g) = \sup_{t \in I} \rho(f(t), g(t))$. Let $d(f, g)$ denote the Fréchet distance between f and g. d defines an equivalence relation $f \sim g$ if and only if $d(f, g) = 0$. Denote by Γ the set of all the equivalence classes called curves and by $[f]$ the class containing f. Γ form a metric space with metric ρ defined by $\rho([f], [g]) = d(f, g)$. The association $f \to [f]$ defines a continuous map $p: \Omega \to \Gamma$. It is proved that Ω is a fibre space over Γ with p as projection. The fibres of Ω are all contractible to a point in itself. There is a natural cross-section $q: \Gamma \to \Omega$ which is isometric. If Γ is imbedded into Ω by means of q, then Γ is a deformation retract of Ω. The last assertion is known to E. Pitcher when A and B are single points. This enables one to compute the homology and homotopy invariants of Γ and its subsets by means of those of Ω and its corresponding subsets. (Received October 11, 1950.)

79*t*. S. T. Hu: *Paths of class C^k in a differentiable manifold.*

Let M be a compact differentiable manifold of class C^r and A, B be two nonvacuous subsets of M. Consider the space Ω of the paths (p-curves) $f: I \to M$ with $f(0) \in A$ and $f(1) \in B$ provided with the compact-open topology. Denote by Ω^k, $k \leqq r$, the subspace of Ω which consists of the paths f of class C^k. It is proved that the spaces Ω and Ω^k have the same homology and homotopy groups. More precisely, the singular polytope (equal to the topologized singular complex) $P(\Omega^k)$ of Ω^k is a deformation retract of the singular polytope $P(\Omega)$ of Ω. (Received October 11, 1950.)

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