homogeneous coordinates really mean geometrically? In the last chapter the start is made at the other end. A point is defined by a set of coordinates and a line by a linear equation; it is verified that this system obeys all the axioms, and the general collineation and correlation are expressed as linear transformations.

It will be seen from the foregoing that the work is severely and carefully argued from beginning to end, and that within its limitations to the real field and to two dimensions it covers just about everything that one could think of including. The shelving of the serious discussion of continuity to a late stage, by assuming one of its chief results as a temporary axiom, probably makes greatly for the intelligibility of the book to beginners. A great number of admirably clear diagrams (probably more than one to every page on an average) illustrate the ideas. The proofs are lucid, and in nearly every case lay bare the fundamental ideas that are being used rather than obscuring these in a mass of detail. The whole book, indeed, is most readable; there are interesting historical notes on the genesis of the ideas presented and a very good bibliography. An appendix of only a couple of pages briefly indicates the nature of the step from real to complex geometry.

PATRICK DUVAL

Extrapolation, interpolation, and smoothing of stationary time series with engineering applications. By Nobert Wiener. Cambridge, Technology Press of Massachusetts Institute of Technology, and New York, Wiley, 1949. 10+163 pp. $4.00.

This is the second book by Professor Wiener on time series and communication engineering published since 1948. While the first book, *Cybernetics*, treated the subject from a general standpoint and was more philosophical than mathematical, the present book is more technical than theoretical, and is intended to give a useful tool for engineers working in the field of electrical communication and related subjects. This book is essentially a reproduction of a pamphlet which had limited circulation during the war.

The main problem discussed in this book is the following: Let $\{x_t\}$ be a stationary time series of class L^2, where the parameter t runs through all integers (discrete case) or all real numbers (continuous case); given a random variable y of class L^2 and a set T of the values of the parameter t, how can we approximate y by finite linear combinations of x_t with t from T?

In the terminology of Hilbert space, this problem can be formulated in a different manner. Let $\{x_t\}$ be a "sequence" (discrete case) or a "curve" (continuous case) in a Hilbert space $\mathfrak{H}$ such that the inner product (x_s, x_t) depends only on $s-t$. This is equivalent to saying that there exists a unitary transformation U defined on $\mathfrak{H}$ and an element x_0 of $\mathfrak{H}$ such that $U^t x_0 = x_t$ for all integers t (discrete case), or there exists a one-parameter group $\{U^t\}$ of unitary transformations U^t defined on $\mathfrak{H}$ and an element x_0 of $\mathfrak{H}$ such that $U^t x_0 = x_t$ for all real numbers t (continuous case). For any set T of the values of the parameter t, let $\mathfrak{M}_T$ be a closed linear subspace of $\mathfrak{H}$ spanned by $\{x_t | t \in T\}$.

Given an element y of $\mathfrak{H}$, the main problems are: (I) how can we compute the distance $d(y, \mathfrak{M}_T)$ of y from $\mathfrak{M}_T$ and (II) how can we find an element y_T of $\mathfrak{M}_T$ which attains or approximates this minimum distance to y? If $t_1 < t_0$, $T = \{s | -\infty < s \leqq t_1\}$ and $y = x_{t_0}$, then this is nothing but a problem of prediction (that is, stochastic extrapolation). If $t_1 < t_0 < t_2$, $T = \{s | -\infty < s \leqq t_1 \text{ or } t_2 \leqq s < \infty\}$ and $y = x_{t_0}$, then this is reduced to a problem of stochastic interpolation. It is easy to see that $d(y, \mathfrak{M}_T) = d(h)$ depends only on $h = t_0 - t_1$ in the first case, and that $d(y, \mathfrak{M}_T) = d(h_1, h_2)$ depends only on $h_1 = t_0 - t_1$ and $h_2 = t_2 - t_0$ in the second case. Further, if $x_t = x_t' + x_t''$, where $\{x_t'\}$ and $\{x_t''\}$ are stationary time series, and if $T = \{s | -\infty < s \leqq t_1\}$, $y = x'_{t_0}$, then this becomes a problem of filtering, that is, a problem of estimating the future, present, or past values of x'_{t_0}, according as $t_0 - t_1$ is >0, $=0$, or <0, when the past values of x_t up to present ($t = t_0$) are known. If we consider x_t' as a real message, x_t'' as random noise, and $x_t = x_t' + x_t''$ as actual information, then this is a problem of eliminating unnecessary noise x_t'' from the information x_t.

Let us consider the continuous case. In this case $\phi(t) = (x_t, x_0) = (U^t x_0, x_0)$ is a positive definite function, and hence, by Bochner's theorem, there exists a distribution function $F(\lambda)$ defined on the infinite interval such that

$$\phi(t) = \int_{-\infty}^{\infty} e^{it\lambda} dF(\lambda). \tag{1}$$

It is then easy to see that the problem of extrapolation and interpolation is equivalent to the problem of finding a trigonometrical polynomial $P(\lambda)$ which attains or approximates the infimum of

$$\int_{-\infty}^{\infty} | e^{it_0\lambda} - P(\lambda) |^2 dF(\lambda), \tag{2}$$

where $P(\lambda) = \sum_{k=1}^{n} a_k e^{it_k\lambda}$ is any finite linear combination of $e^{it_k\lambda}$ with $t_k \in T$, $k = 1, \cdots, n$.

In this book the discussion is limited to the case when $F(\lambda)$ is absolutely continuous. It turns out that either $d(h)>0$ for all $h>0$ or $d(h)\equiv 0$ for all $h>0$, and the first case happens if and only if

$$\int_{-\infty}^{\infty} \frac{\log F'(\lambda)}{1+\lambda^2}\, d\lambda > -\infty. \tag{3}$$

Here the problem is further reduced to the discussion of the possibility, and the concrete method, of representing $F'(\lambda)$ as a product of two functions $\Phi_+(\lambda)$ and $\Phi_-(\lambda)$ which are the boundary values on the real line of functions $\Phi_+(\lambda+i\mu)$ and $\Phi_-(\lambda+i\mu)$ defined and analytic in the upper and the lower half plane, respectively.

These problems are not only mathematically interesting, but also extremely important in application. This book gives a detailed discussion of the theory of approximation of this type which has been developed by the author himself. As is pointed out by the author, the theory of extrapolation and interpolation for the discrete case has been investigated independently by A. Kolmogoroff (Bull. Acad. Sci. USSR, Ser. Math. vol. 5 (1941) pp. 3–14) by using the method of operator theory in Hilbert space. It is easy to see that in the discrete case $F(\lambda)$ becomes a distribution defined on the set of real numbers mod. 1, or equivalently on the unit circle. Kolmogoroff does not assume that $F(\lambda)$ is absolutely continuous. In case $F(\lambda)$ is absolutely continuous, the problem is again reduced to that of finding a representation of $F'(\lambda)$ as a product of two functions which are the boundary values on the unit circle of functions defined and analytic inside and outside the unit circle, respectively.

It is to be remarked that Kolmogoroff is mainly interested in problem (I), that is, in the problem of finding the exact values of $d(h)$ and $d(h_1, h_2)$ in terms of h, h_1, h_2, and $F(\lambda)$, while the author is more interested in problem (II), that is, in the problem of finding a concrete method of obtaining a linear combination which attains or approximates the minimum distance. The method of the author is straight forward and is based on Fourier analysis. This fact makes it possible to solve practical prediction problems experimentally in many important cases by constructing a suitable electrical network. It is remarkable to note that the theory of generalized harmonic analysis developed by the author some twenty years ago is exactly the right tool for this purpose.

The whole treatment is based on the author's theory of generalized harmonic analysis (Acta Math. vol. 55 (1930) pp. 117–258) and as in many other writings of the author, technical terms from electrical engineering are often used to explain the intuitive meaning of various mathematical concepts. These technical terms are very useful and illustrative in view of its application to practical problems. On the other hand, it seems that there are many places in this book in which geometrical language from Hilbert space could have been successfully introduced to clarify the situation. Also, the interpretation of each $x_t = x_t(\omega)$ as a random variable, and the observation of the fact that, because the ergodicity of the system in case $F(\lambda)$ is absolutely continuous, the "time average"

$$\lim_{A\to\infty} \frac{1}{2A} \int_{-A}^{A} x_{s+u}(\omega) x_{t+u}(\omega) du \tag{4}$$

is equal to the "space average"

$$(x_s, x_t) = \int x_s(\omega) x_t(\omega) d\omega \tag{5}$$

would have simplified the argument substantially.

The book consists of an introduction, five chapters, and three appendices. After explaining the general outline of the problem in the introduction, the author gives in Chapter I a review of generalized harmonic analysis which is necessary for the understanding of the following chapters. Chapters II and III are devoted to the problems of prediction and filtering respectively. In Chapter IV there is given a brief account of the theory of multiple prediction, that is, the theory of prediction when we deal with more than one time series at the same time. This theory is still incomplete, and the problem analogous to that of factorization of $F'(\lambda)$ is not yet solved. Finally, in Chapter V there is given a short discussion on the application of similar methods to a problem of approximate differentiation.

The book concludes with three appendices: a numerical table of Laguerre functions which are useful in practical application, and two papers by N. Levinson reprinted from Journal of Mathematics and Physics vol. 25 (1946) pp. 261–278, vol. 26 (1947) pp. 110–119. The first of these notes by N. Levinson gives a practical method of eliminating random noise (filtering) in communication engineering by using electrical networks, and the second note is an excellent, heuristic but very clear, exposition which explains the structure of Wiener's theory of prediction and filtering.

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